

Widths of weighted Sobolev classes with restrictions $f(a) = \dots = f^{(k-1)}(a) = f^{(k)}(b) = \dots = f^{(r-1)}(b) = 0$ and spectra of non-linear differential equations

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1 Introduction

In papers of Pinkus [23], Buslaev and Tikhomirov [5] it was proved that the Kolmogorov widths of the Sobolev class $W_p^r[a, b]$ with some boundary conditions in the space $L_q[a, b]$ for $p \geq q$ coincide with spectral numbers of some special non-linear differential equation with the same boundary conditions. For weighted Sobolev classes with piecewise continuous weights such theorems were obtained by Buslaev [3], [4], and for $r = 1$ and $g \in L_{p'}[a, b]$, $v \in L_q[a, b]$, by Edmunds and Lang [9].

Buslaev and Tikhomirov [5] considered the following boundary conditions for functions x : 1) $x^{(j)}(a) = 0$, $0 \leq j \leq r-1$, 2) $x^{(j)}(a) = 0$, $x^{(j)}(b) = 0$, $0 \leq j \leq r-1$, 3) no boundary conditions, 4) $x^{(j)}(a) = 0$ for even j , $x^{(j)}(b) = 0$ for odd j , $0 \leq j \leq r-1$, 5) periodic conditions. In case 1) this result was generalized in [30] for weighted Sobolev classes in weighted Lebesgue space. It was supposed that the corresponding two-weighted Riemann – Liouville operator (see formula (3)) is compact. Moreover, it was proved that the widths always strictly decrease. Here we obtain the analogue of this theorem for the boundary conditions $x(a) = \dots = x^{(k-1)}(a) = x^{(k)}(b) = \dots = x^{(r-1)}(b) = 0$.

We give the definition of the Kolmogorov, linear, Gelfand and Bernstein widths.

Let $(X, \|\cdot\|_X)$ be a normed space, let X^* be its dual, and let $\mathcal{L}_n(X)$, $n \in \mathbb{Z}_+$, be the family of subspaces of X of dimension at most n . Denote by $L(X, Y)$ the space of continuous linear operators from X into a normed space Y . Also, by $\text{rk } A$ denote the dimension of the image of an operator $A \in L(X, Y)$, and by $\|A\|_{X \rightarrow Y}$, its norm.

By the Kolmogorov n -width of an absolutely convex set $M \subset X$ in the space X , we mean the quantity

$$d_n(M, X) = \inf_{L \in \mathcal{L}_n(X)} \sup_{x \in M} \inf_{y \in L} \|x - y\|_X,$$

by the linear n -width, the quantity

$$\lambda_n(M, X) = \inf_{A \in L(X, X), \text{rk } A \leq n} \sup_{x \in M} \|x - Ax\|_X,$$

by the Gelfand n -width, the quantity

$$d^n(M, X) = \inf_{x_1^*, \dots, x_n^* \in X^*} \sup\{\|x\| : x \in M, x_j^*(x) = 0, 1 \leq j \leq n\},$$

and by the Bernstein n -width, the quantity

$$b_n(M, X) = \sup_{\dim L = n+1} \sup\{R > 0 : B_R(0) \cap L \subset M\}$$

(here the supremum is taken over the family of linear $n + 1$ -dimensional subspaces). The following relations between widths are well-known (here M is an algebraic sum of a compact and a finite dimensional subspace; see [26, p. 207]):

$$b_n(M, X) \leq d_n(M, X) \leq \lambda_n(M, X), \quad b_n(M, X) \leq d^n(M, X) \leq \lambda_n(M, X). \quad (1)$$

In [21] a notion of strict s -numbers of a linear continuous operator was introduced. Examples of strict s -numbers are the Kolmogorov numbers, the Gelfand numbers, the approximation numbers and the Bernstein numbers. The Kolmogorov numbers of an operator $A \in L(X, Y)$ are defined by

$$d_n(A : X \rightarrow Y) = d_n(A(B_X), Y),$$

the approximation numbers, by

$$a_n(A : X \rightarrow Y) = \inf\{\|A - A_n\|_{X \rightarrow Y} : \text{rk } A_n \leq n\},$$

the Gelfand numbers, by

$$c_n(A : X \rightarrow Y) = \inf_{\{x_j^*\}_{j=1}^n \subset X^*} \sup\{\|Ax\| : x \in \cap_{j=1}^n \ker x_j^*, \|x\| \leq 1\},$$

the Bernstein numbers, by

$$b_n(A : X \rightarrow Y) = \sup_{\dim L \geq n+1} \sup\{R > 0 : B_R(0) \subset A(B_X \cap L)\}$$

(the supremum is taken over the family of subspaces in X of dimension $n + 1$ or larger). Then $b_n(A : X \rightarrow Y) = b_n(A(B_X), Y)$. Heinrich [13] proved that if the operator A is a compact embedding, then

$$a_n(A : X \rightarrow Y) = \lambda_n(A(B_X), Y), \quad c_n(A : X \rightarrow Y) = d^n(A(B_X), Y). \quad (2)$$

Let $r \in \mathbb{N}$. For measurable nonnegative functions u, w and $m \in \mathbb{N}$ we define the Riemann – Liouville operators

$$\begin{aligned} \tilde{I}_{m,u,w,b}\varphi(t) &= \frac{w(t)}{(m-1)!} \int_t^b (s-t)^{m-1} u(s) \varphi(s) ds, \\ I_{m,u,w,a}\psi(t) &= \frac{w(t)}{(m-1)!} \int_a^t (t-s)^{m-1} u(s) \psi(s) ds. \end{aligned} \quad (3)$$

If $m = 1$, they are called two-weighted Hardy-type operators. Given $r \geq 2$, $1 \leq m \leq r - 1$, we set

$$\tilde{I}_{r,u,w}^{a,b,m} = I_{m,1,w,a} \circ \tilde{I}_{r-m,u,1,b}, \quad \tilde{I}_{r,u,w}^m = \tilde{I}_{r,u,w}^{0,1,m}. \quad (4)$$

The criterion of continuity of the operator $\tilde{I}_{r,u,w}^{a,b,m} : L_p[a, b] \rightarrow L_q[a, b]$ was obtained by Heinig and Kufner [16]. The criterion of continuity of two-weighted Riemann – Liouville operators $I_{m,u,w,a} : L_p[a, b] \rightarrow L_q[a, b]$ was obtained by Stepanov [24, 25].

Given $1 < p < \infty$, we set $p' = \frac{p}{p-1}$. Then, if the operator $\tilde{I}_{r,u,w}^{a,b,m} : L_p[a, b] \rightarrow L_q[a, b]$ is continuous, then its dual has the form $\tilde{I}_{r,w,u}^{a,b,r-m} : L_{q'}[a, b] \rightarrow L_{p'}[a, b]$.

Let $g, v : [a, b] \rightarrow (0, \infty)$ be measurable functions, $1 < p < \infty$, $1 < q < \infty$, $r \geq 2$, $1 \leq k \leq r - 1$, and let the operator $\tilde{I}_{r,g,v}^{a,b,k} : L_p[a, b] \rightarrow L_q[a, b]$ be continuous. Then Heinig and Kufner's criterion [16, Theorem 1 and Section 2] implies that for any $\varepsilon \in (0, 1)$

$$\begin{aligned} g &\in L_{p'}(a + \varepsilon, b], \quad v \in L_q(a + \varepsilon, b], \\ (s \mapsto (s - a)^k v(s)) &\in L_q[a, b - \varepsilon], \quad (s \mapsto (s - a)^{r-k} g(s)) \in L_{p'}[a, b - \varepsilon]. \end{aligned} \quad (5)$$

We set

$$W_{p,g}^{r,k}[a, b] = \{\tilde{I}_{r,g,1}^{a,b,k} \varphi : \|\varphi\|_{L_p[a,b]} \leq 1\}. \quad (6)$$

From (5) it follows that the iterated integral

$$\frac{v(t)}{(k-1)!(r-k-1)!} \int_a^t (t-s)^{k-1} \int_s^b (\tau-s)^{r-k-1} g(\tau) \varphi(\tau) d\tau ds$$

is well-defined for all $\varphi \in L_p[a, b]$. We call the set $W_{p,g}^{r,k}[a, b]$ a weighted Sobolev class with restrictions $f(a) = \dots = f^{(k-1)}(a) = f^{(k)}(b) = \dots = f^{(r-1)}(b) = 0$.

Let

$$\|f\|_{L_{q,v}[a,b]} = \|vf\|_{L_q[a,b]}, \quad L_{q,v}[a, b] = \{f : \|f\|_{L_{q,v}[a,b]} < \infty\}.$$

We call $L_{q,v}[a, b]$ a weighted Lebesgue space.

We set $\mathcal{W}_{p,g}^{r,k}[a, b] = \text{span } W_{p,g}^{r,k}[a, b]$ and endow this space with the norm $\|f\|_{\mathcal{W}_{p,g}^{r,k}[a,b]} = \left\| \frac{f(\tau)}{g} \right\|_{L_p[a,b]}$.

Throughout we assume that $1 < q \leq p < \infty$, $g(t) > 0$, $v(t) > 0$ a.e. and the operator $\tilde{I}_{r,g,v}^{a,b,k} : L_p[a, b] \rightarrow L_q[a, b]$ is compact.

We denote

$$h_{(\sigma)} = |h|^{\sigma-1} \text{sgn } h$$

and consider the following boundary value problem:

$$\begin{cases} x^{(r)} = (-1)^{r-k} g^{p'} y_{(p')}, \\ y^{(r)} = (-1)^k \theta^q v^q x_{(q)}, \\ x(a) = \dots = x^{(k-1)}(a) = x^{(k)}(b) = \dots = x^{(r-1)}(b) = 0, \\ y(a) = \dots = y^{(r-k-1)}(a) = y^{(r-k)}(b) = \dots = y^{(r-1)}(b) = 0, \\ \left\| \frac{x^{(r)}}{g} \right\|_{L_p[a, b]} = 1; \end{cases} \quad (7)$$

here $\theta > 0$, the functions $x, \dots, x^{(k-1)}, y, \dots, y^{(r-k-1)}$ are locally absolutely continuous on $[a, b]$, and the functions $x^{(k)}, \dots, x^{(r-1)}, y^{(r-k)}, \dots, y^{(r-1)}$ are locally absolutely continuous on $(a, b]$.

Let f be a measurable function, $n \in \mathbb{Z}_+$. We say that f has exactly n points of sign change if there are points $a = t_0 < t_1 < \dots < t_n < t_{n+1} = b$ such that

1. for any $1 \leq j \leq n$, $t, s \in [t_{j-1}, t_j]$ the inequality $f(t)f(s) \geq 0$; in addition, the set $\{t \in [t_{j-1}, t_j] : f(t) \neq 0\}$ has a positive measure for each $j \in \{1, \dots, n\}$;
2. for any $1 \leq j \leq n-1$, $t \in [t_{j-1}, t_j]$, $s \in [t_j, t_{j+1}]$ the inequality $f(t)f(s) \leq 0$ holds.

We say that f has no more than n points of sign change if f has exactly m points of sign change with $0 \leq m \leq n$.

Let $x \in \mathcal{W}_{p,g}^r[a, b]$, $y \in \mathcal{W}_{q,v}^r[a, b]$, $\theta > 0$, $n \in \mathbb{Z}_+$. We say that $(x, y, \theta) \in SP_n$ (correspondingly, $(x, y, \theta) \in \widetilde{SP}_n$), if (7) holds and the function x has exactly n (correspondingly, no more than n) points of sign change; we call x the *spectral function*. Denote by sp_n (correspondingly, by \widetilde{sp}_n) the set of numbers $\theta > 0$ such that $(x, y, \theta) \in SP_n$ (correspondingly, $(x, y, \theta) \in \widetilde{SP}_n$) for some $x \in \mathcal{W}_{p,g}^r[a, b]$, $y \in \mathcal{W}_{q,v}^r[a, b]$. We set $\bar{\theta}_n = \sup sp_n$, $\tilde{\theta}_n = \sup \widetilde{sp}_n$.

Theorem 1. *Let $1 < q \leq p < \infty$, $g(t) > 0$, $v(t) > 0$ a.e., the operator $\tilde{I}_{r,g,v}^{a,b,k} : L_p[a, b] \rightarrow L_q[a, b]$ is compact. Then*

$$d_n(W_{p,g}^{r,k}[a, b], L_{q,v}[a, b]) = \lambda_n(W_{p,g}^{r,k}[a, b], L_{q,v}[a, b]) = d^n(W_{p,g}^{r,k}[a, b], L_{q,v}[a, b]) = \bar{\theta}_n^{-1}. \quad (8)$$

Moreover, the widths strictly decrease with respect to n . If $p = q$, then the set sp_n is a singleton and

$$b_n(W_{p,g}^{r,k}[a, b], L_{p,v}[a, b]) = d_n(W_{p,g}^{r,k}[a, b], L_{p,v}[a, b]) = \bar{\theta}_n^{-1}. \quad (9)$$

The equality (9) is proved similarly as in [30, p. 392–393] and [23, p. 24–25]. Notice that in the paper of Edmunds and Lang [10] the coincidence of all strict s -numbers of a compact two-weighted Hardy operator was proved for $p = q$.

The paper is organized as follows. In Section 2 we apply the method from the paper of Buslaev and Tikhomirov [5] and prove (8) with $\tilde{\theta}_n$ instead of $\bar{\theta}_n$. In proof

of the upper estimate we use the integration operator with the kernel $G(t, \tau, \xi, \eta)$ defined by formula (25), which differ from the kernel from [5].

In Sections 3–5 we prove that the widths are strictly decreasing. The method of the proof is the same as in [30]. From the strict monotonicity of widths we now obtain (8) with $\bar{\theta}_n$. Everywhere in §2–5 we write only the fragments of proof that differ from arguments in [5] and [30]. Notice that everywhere in §2–5 without loss of generality we may assume that $[a, b] = [0, 1]$.

In Section 6 we apply the main result in estimating the spectral numbers for (7) with weights $g(x) = x^{-\beta_g} |\ln x|^{-\alpha_g} \rho_g(|\ln x|)$, $v(x) = x^{-\beta_v} |\ln x|^{-\alpha_v} \rho_v(|\ln x|)$; here ρ_g, ρ_v are “slowly varying” functions, $\alpha_g + \alpha_v = r + \frac{1}{q} - \frac{1}{p}$. The order estimates immediately follow from (8) and results of the paper [31]. In addition, if the parameters are such that $gv \in L_\varkappa$ with $\frac{1}{\varkappa} = r + \frac{1}{q} - \frac{1}{p}$, then asymptotics of widths and spectral numbers is obtained.

For piecewise continuous weights asymptotics of widths for $p \geq q$ was obtained by Buslaev [4], and for $g \in L_{p'}[a, b]$, $v \in L_q[a, b]$, $r = 1$, by Edmunds and Lang [8] (see also Buslaev’s paper [3] for $r \in \mathbb{N}$ and some supplementary conditions on weights). For $p = q$, $r = 1$ and more general conditions on weights asymptotics was obtained by Evans, Harris, Lang, Edmunds and Kerman in [6, 7, 17], and for $p \geq q$, $r \in \mathbb{N}$, in [32].

The problem of determining exact values of widths was studied in papers of Kolmogorov, Tikhomirov, Babadzanov, Makovoz, Ligun, Pinkus and others [15, 18, 19, 22, 27–29] (for details, see [5]). In paper of Malykhin [20] the asymptotics of

$$d_{n+r}(W_\infty^r[-1, 1], C[-1, 1])/d_r(W_\infty^r[-1, 1], C[-1, 1]), \quad r \rightarrow \infty,$$

was obtained. In papers of Babenko and Parfinovich [1, 2] the subspaces of splines of different defect were studied as extremal subspaces in the problem of calculating widths.

2 The application of Buslaev and Tikhomirov’s method

We may assume that $[a, b] = [0, 1]$.

First we prove that

$$d_n(W_{p,g}^{r,k}[0, 1], L_{q,v}[0, 1]) \geq \tilde{\theta}_n^{-1}, \quad d^n(W_{p,g}^{r,k}[0, 1], L_{q,v}[0, 1]) \geq \tilde{\theta}_n^{-1}. \quad (10)$$

This together with (1) yield that the similar lower estimate holds for linear widths.

The system (7) with $[a, b] = [0, 1]$ can be written as the system of integral equations:

$$\begin{cases} x = \tilde{I}_{r,q,1}^k(g^{p'-1}y_{(p')}), \\ y = \theta^q \tilde{I}_{r,v,1}^{r-k}(v^{q-1}x_{(q)}), \\ \|g^{p'-1}y_{(p')}\|_{L_p[0,1]} = 1. \end{cases} \quad (11)$$

Notice that $\|g^{p'-1}y_{(p')}\|_{L_p[0,1]} = \|gy\|_{L_{p'}[0,1]}$.

Let u be a piecewise continuous function with values ± 1 . The Buslaev's iteration process [5, §6] is written as follows:

$$x_0(\cdot, u) = \tilde{I}_{r,g,1}^k u(\cdot);$$

for $m \in \mathbb{N}$ we set

$$y_m(\cdot, u) = \theta_{m-1}^q(u) \tilde{I}_{r,v,1}^{r-k}[(vx_{m-1})_{(q)}], \quad (12)$$

$$x_m(\cdot, u) = \tilde{I}_{r,g,1}^k[(gy_m)_{(p')}] \quad (13)$$

here $\theta_{m-1}(u) > 0$ is chosen so that

$$\|gy_m\|_{L_{p'}[0,1]} = 1 \quad (14)$$

(it is possible, since $v(\cdot)x_{m-1}(\cdot, u) \neq 0$ a.e., which can be easily proved by induction). Since the operator $\tilde{I}_{r,g,v}^k : L_p[0, 1] \rightarrow L_q[0, 1]$ and its dual $\tilde{I}_{r,v,g}^{r-k} : L_{q'}[0, 1] \rightarrow L_{p'}[0, 1]$ are compact, by induction method it can be proved that $x_m(\cdot, u) \in L_{q,v}[0, 1]$. Indeed, $u \in L_p[0, 1]$. Therefore, $x_0(\cdot, u) \in L_{q,v}[0, 1]$. Let $x_{m-1}(\cdot, u) \in L_{q,v}[0, 1]$. Then

$$\begin{aligned} (v(\cdot)x_{m-1}(\cdot, u))_{(q)} \in L_{q'}[0, 1] &\Rightarrow g(\cdot)y_m(\cdot, u) \in L_{p'}[0, 1] \Rightarrow \\ &\Rightarrow (g(\cdot)y_m(\cdot, u))_{(p')} \in L_p[0, 1] \Rightarrow x_m(\cdot, u) \in L_{q,v}[0, 1]. \end{aligned} \quad (15)$$

The following lemma is similar to Lemma 1 from [5, §6].

Lemma 1. *The following estimates hold:*

$$\|x_{m-1}(\cdot, u)\|_{L_{q,v}[0,1]} \leq [\theta_{m-1}(u)]^{-1} \leq \|x_m(\cdot, u)\|_{L_{q,v}[0,1]}.$$

Proof. The arguments are almost similar as in [5], but instead of integration by parts we apply the Fubini theorem. Everywhere we denote $x_j = x_j(\cdot, u)$, $\theta_j = \theta_j(u)$. We have

$$1 = \int_0^1 \left| \frac{x_m^{(r)}}{g} \right|^p dt \stackrel{(13),(14)}{=} \int_0^1 \frac{x_m^{(r)}}{g} \left(\frac{x_m^{(r)}}{g} \right)_{(p)} dt \stackrel{(13)}{=} \int_0^1 (-1)^{r-k} \frac{x_m^{(r)}}{g} gy_m dt =: I.$$

Let

$$\varphi = \frac{x_m^{(r)}}{g} \stackrel{(13)}{=} (gy_m)_{(p')}, \quad \psi = v^{q-1}[x_{m-1}]_{(q)}. \quad (16)$$

Then by (15) we get

$$\varphi \in L_p[0, 1], \quad \psi \in L_{q'}[0, 1], \quad gy_m \in L_{p'}[0, 1],$$

$$\begin{aligned}
& I \stackrel{(12)}{=} [\theta_{m-1}]^q \int_0^1 \varphi \cdot \tilde{I}_{r,v,g}^{r-k} \psi \, dt \stackrel{(3),(4)}{=} \\
& = \frac{[\theta_{m-1}]^q}{(k-1)!(r-k-1)!} \int_0^1 \varphi(t) g(t) \int_0^t (t-s)^{r-k-1} \int_s^1 (\tau-s)^{k-1} v(\tau) \psi(\tau) \, d\tau \, ds \, dt = \\
& = \frac{[\theta_{m-1}]^q}{(k-1)!(r-k-1)!} \int_0^1 \psi(\tau) v(\tau) \int_0^\tau (\tau-s)^{k-1} \int_s^1 (t-s)^{r-k-1} g(t) \varphi(t) \, dt \, ds \, d\tau \stackrel{(3),(4)}{=} \\
& = [\theta_{m-1}]^q \int_0^1 \psi \cdot \tilde{I}_{r,g,v}^k \varphi \, d\tau \stackrel{(13),(16)}{=} [\theta_{m-1}]^q \int_0^1 v^{q-1} [x_{m-1}]_{(q)} \cdot v x_m \, dt \leq \\
& \leq [\theta_{m-1}]^q \|x_{m-1}\|_{L_{q,v}[0,1]}^{q-1} \|x_m\|_{L_{q,v}[0,1]}.
\end{aligned}$$

Thus,

$$1 \leq [\theta_{m-1}]^q \|x_{m-1}\|_{L_{q,v}[0,1]}^{q-1} \|x_m\|_{L_{q,v}[0,1]}. \quad (17)$$

Further,

$$1 \stackrel{(13),(14)}{=} \left\| \frac{x_{m-1}^{(r)}}{g} \right\|_{L_p[0,1]} \left\| \frac{x_m^{(r)}}{g} \right\|_{L_p[0,1]}^{p-1} \geq \int_0^1 \left(\frac{x_m^{(r)}}{g} \right)_{(p)} \frac{x_{m-1}^{(r)}}{g} \, dt \stackrel{(13)}{=} \int_0^1 g y_m \frac{x_{m-1}^{(r)}}{g} \, dt =: J.$$

Applying (12) together with the Fubini theorem, we get that

$$J = [\theta_{m-1}]^q \int_0^1 v^{q-1} [x_{m-1}]_{(q)} \cdot \tilde{I}_{r,g,v}^k \frac{x_{m-1}^{(r)}}{g} \, dt = [\theta_{m-1}]^q \|x_{m-1}\|_{L_{q,v}[0,1]}^q.$$

Hence, $\|x_{m-1}\|_{L_{q,v}[0,1]} \leq [\theta_{m-1}]^{-1}$. This together with (17) yields that $[\theta_{m-1}]^{-1} \leq \|x_m\|_{L_{q,v}[0,1]}$. \square

Repeating arguments from [5] (see Lemma 2 from Section 6 and its corollary, as well as Lemma 1 from Section 8), we get the following assertions.

1. One can choose a subsequence of $\{x_m(\cdot, u)\}_{m \in \mathbb{N}}$ convergent to a spectral function in the space $L_{q,v}[0, 1]$.
2. The set SP_0 is non-empty.
3. The estimates (10) hold.

Now we prove the inequality

$$\lambda_n(W_{p,g}^{r,k}[0, 1], L_{q,v}[0, 1]) \leq \tilde{\theta}_n^{-1}. \quad (18)$$

From (1) it follows that the same inequality holds for Kolmogorov and Gelfand widths.

Similarly as in [30, p. 367] we prove that the set $\tilde{s}p_n$ is closed.

Let $(\bar{x}, \bar{y}, \bar{\theta}) \in SP_m$, $0 \leq m \leq n$. We show that

$$\lambda_n(W_{p,g}^{r,k}[0, 1], L_{q,v}[0, 1]) \leq \bar{\theta}^{-1}. \quad (19)$$

Since the operator $\tilde{I}_{r,g,v}^k : L_p[0, 1] \rightarrow L_q[0, 1]$ is compact, by (6) and Heinrich's results (2) it is sufficient to prove that

$$a_n(\tilde{I}_{r,g,v}^k : L_p[0, 1] \rightarrow L_q[0, 1]) \leq \bar{\theta}^{-1}$$

(recall that a_n are the approximation numbers).

Applying the Rolle theorem and repeating arguments from Proposition 1 in [30] (see also [5, p. 1594]), we obtain the following assertion.

Proposition 1. *Let $(\bar{x}, \bar{y}, \bar{\theta}) \in SP_m$, $m \in \mathbb{Z}_+$. Then the following assertions hold.*

1. *The functions \bar{x} , \bar{y} and their derivatives $\bar{x}^{(j)}$, $\bar{y}^{(j)}$, $1 \leq j \leq r-1$, have exactly m zeros on $(0, 1)$.*

2. *Let*

$$0 < \xi_1 < \dots < \xi_m < 1, \quad 0 < \eta_1 < \dots < \eta_m < 1, \quad \bar{x}(\xi_i) = 0, \quad \bar{y}(\eta_i) = 0, \quad 1 \leq i \leq m.$$

Then ξ_i are points of sign change of the function \bar{x} , η_i are points of sign change of the function \bar{y} .

3. *We have $\dot{\bar{x}}(\xi_i) \neq 0$, $\dot{\bar{y}}(\eta_i) \neq 0$, $1 \leq i \leq m$.*

Now we obtain the alternation condition for points ξ_i , η_i .

Proposition 2. *Let the points ξ_i , η_i ($1 \leq i \leq m$) be such as in Proposition 1. Then*

$$\eta_{j+k-r} < \xi_j < \eta_{j+k}. \quad (20)$$

Proof. From the Rolle theorem and the condition $\bar{x}(0) = \dots = \bar{x}^{(k-1)}(0) = 0$ it follows that the function $\bar{x}^{(k)}$ has m zeroes $0 < \mu_1 < \dots < \mu_m < 1$ on the interval $(0, 1)$; moreover, the alternation condition $\mu_j < \xi_j < \mu_{j+k}$ holds. By (7), the points of sign change of the function $\bar{x}^{(r)}$ coincide with zeroes of the function \bar{y} on $(0, 1)$. By the Rolle theorem and the condition $\bar{x}^{(k)}(1) = \dots = \bar{x}^{(r-1)}(1) = 0$, the alternation condition $\mu_j < \eta_j < \mu_{j+r-k}$ holds. Hence, $\xi_j < \mu_{j+k} < \eta_{j+k}$, $\xi_j > \mu_j > \eta_{j+k-r}$. This completes the proof of (20). \square

Definition of the kernel $G(t, \tau, \xi, \eta)$. Let $\mu = \{\mu_j\}_{j=1}^J \subset [0, 1]$, $\nu = \{\nu_j\}_{j=1}^J \subset [0, 1]$, $l \in \mathbb{N}$. We set

$$K_l(\mu, \nu) = \begin{vmatrix} (\mu_1 - \nu_1)_+^{l-1} & (\mu_2 - \nu_1)_+^{l-1} & \dots & (\mu_J - \nu_1)_+^{l-1} \\ (\mu_1 - \nu_2)_+^{l-1} & (\mu_2 - \nu_2)_+^{l-1} & \dots & (\mu_J - \nu_2)_+^{l-1} \\ \dots & \dots & \dots & \dots \\ (\mu_1 - \nu_J)_+^{l-1} & (\mu_2 - \nu_J)_+^{l-1} & \dots & (\mu_J - \nu_J)_+^{l-1} \end{vmatrix} \quad (21)$$

(if $l = 1$, then we set $x_+^0 = 0$ for $x < 0$ and $x_+^0 = 1$ for $x \geq 0$). In [14, Lemma 9.2] it was proved that if $\mu_1 < \dots < \mu_J$, $\nu_1 < \dots < \nu_J$, then $K_l(\mu, \nu) \geq 0$; in addition, if the alternation condition $\nu_j < \mu_j < \nu_{j+l}$ holds, then $K_l(\mu, \nu) > 0$.

Let now $r \geq 2$, $1 \leq k \leq r-1$, $\mu = \{\mu_j\}_{j=1}^J \subset [0, 1]$, $\nu = \{\nu_j\}_{j=1}^J \subset [0, 1]$. We set

$$K_{r,k}(\mu, \nu) = \int_{[0,1]^J} K_k(\mu, \alpha) K_{r-k}(\nu, \alpha) d\alpha, \quad (22)$$

where $\alpha = (\alpha_1, \dots, \alpha_J)$.

Lemma 2. *Let $0 < \mu_1 < \dots < \mu_J < 1$, $0 < \nu_1 < \dots < \nu_J < 1$. Then $K_{r,k}(\mu, \nu) \geq 0$. In addition, if the alternation condition*

$$\nu_{j+k-r} < \mu_j < \nu_{j+k}, \quad (23)$$

holds, then $K_{r,k}(\mu, \nu) > 0$.

Proof. There exists a rearrangement $\sigma \in S_J$ such that $\alpha_{\sigma(1)} \leq \dots \leq \alpha_{\sigma(J)}$. We set $\tilde{\alpha} = (\alpha_{\sigma(1)}, \dots, \alpha_{\sigma(J)})$. Then $K_k(\mu, \tilde{\alpha}) \geq 0$, $K_{r-k}(\nu, \tilde{\alpha}) \geq 0$. Rearranging lines in the determinant, we obtain that

$$K_k(\mu, \alpha) K_{r-k}(\nu, \alpha) = (-1)^\sigma K_k(\mu, \tilde{\alpha}) \cdot (-1)^\sigma K_{r-k}(\nu, \tilde{\alpha}) = K_k(\mu, \tilde{\alpha}) \cdot K_{r-k}(\nu, \tilde{\alpha}) \geq 0. \quad (24)$$

It remains to integrate over α .

Now let the alternation condition hold. We show that there exists a sequence $0 < \alpha_1 < \dots < \alpha_J < 1$ such that $\alpha_j < \mu_j < \alpha_{j+k}$, $\alpha_j < \nu_j < \alpha_{j+r-k}$. This together with (24) implies that $K_{r,k}(\mu, \nu) > 0$.

We take a sufficiently small $\varepsilon > 0$ and set $\alpha_j = \min\{\mu_j, \nu_j\} - \varepsilon$. Since $\mu_j < \mu_{j+1}$, $\nu_j < \nu_{j+1}$, we have $\alpha_j < \alpha_{j+1}$ for small $\varepsilon > 0$. The inequalities $\alpha_j < \mu_j$, $\alpha_j < \nu_j$ hold by construction. We show that $\mu_j < \alpha_{j+k}$ and $\nu_j < \alpha_{j+r-k}$. To this end, it is sufficient to check the inequalities $\mu_j < \min\{\mu_{j+k}, \nu_{j+k}\}$, $\nu_j < \min\{\mu_{j+r-k}, \nu_{j+r-k}\}$. They hold by the strict monotonicity of $\{\mu_i\}_{i=1}^J$, $\{\nu_i\}_{i=1}^J$ and the alternation condition (23). \square

Lemma 3. Let $\varphi, \psi \in L_\infty([0, 1]^2)$. For $\mu = \{\mu_j\}_{j=1}^J$, $\nu = \{\nu_j\}_{j=1}^J$, $\alpha = \{\alpha_j\}_{j=1}^J \subset [0, 1]$ we set

$$\Phi(\mu, \alpha) = \begin{vmatrix} \varphi(\mu_1, \alpha_1) & \varphi(\mu_2, \alpha_1) & \dots & \varphi(\mu_J, \alpha_1) \\ \varphi(\mu_1, \alpha_2) & \varphi(\mu_2, \alpha_2) & \dots & \varphi(\mu_J, \alpha_2) \\ \dots & \dots & \dots & \dots \\ \varphi(\mu_1, \alpha_J) & \varphi(\mu_2, \alpha_J) & \dots & \varphi(\mu_J, \alpha_J) \end{vmatrix},$$

$$\Psi(\nu, \alpha) = \begin{vmatrix} \psi(\nu_1, \alpha_1) & \psi(\nu_2, \alpha_1) & \dots & \psi(\nu_J, \alpha_1) \\ \psi(\nu_1, \alpha_2) & \psi(\nu_2, \alpha_2) & \dots & \psi(\nu_J, \alpha_2) \\ \dots & \dots & \dots & \dots \\ \psi(\nu_1, \alpha_J) & \psi(\nu_2, \alpha_J) & \dots & \psi(\nu_J, \alpha_J) \end{vmatrix},$$

$$\Lambda(\mu, \nu) = \int_{[0,1]^J} \Phi(\mu, \alpha) \Psi(\nu, \alpha) d\alpha, \quad H(t, \tau) = \int_0^1 \varphi(t, s) \psi(\tau, s) ds.$$

Then

$$\Lambda(\mu, \nu) = J! \begin{vmatrix} H(\mu_1, \nu_1) & H(\mu_2, \nu_1) & \dots & H(\mu_J, \nu_1) \\ H(\mu_1, \nu_2) & H(\mu_2, \nu_2) & \dots & H(\mu_J, \nu_2) \\ \dots & \dots & \dots & \dots \\ H(\mu_1, \nu_J) & H(\mu_2, \nu_J) & \dots & H(\mu_J, \nu_J) \end{vmatrix}.$$

The assertion is a consequence of the formula (3.12) in [14]. For convenience, we give the proof.

Proof. We have

$$\begin{aligned} \Lambda(\mu, \nu) &= \sum_{\sigma \in S_J} \sum_{\pi \in S_J} (-1)^{\sigma+\pi} \int_{[0,1]^J} \prod_{i=1}^J \varphi(\mu_{\sigma(i)}, \alpha_i) \prod_{j=1}^J \psi(\nu_{\pi(j)}, \alpha_j) d\alpha_1 \dots d\alpha_J = \\ &= \sum_{\sigma \in S_J} \sum_{\pi \in S_J} (-1)^{\sigma+\pi} \prod_{j=1}^J \left(\int_0^1 \varphi(\mu_{\sigma(j)}, \alpha_j) \psi(\nu_{\pi(j)}, \alpha_j) d\alpha_j \right) = \\ &= \sum_{\sigma \in S_J} \sum_{\pi \in S_J} (-1)^{\sigma+\pi} \prod_{j=1}^J H(\mu_{\sigma(j)}, \nu_{\pi(j)}) = \sum_{\sigma \in S_J} \sum_{\pi \in S_J} (-1)^{\sigma+\pi} \prod_{i=1}^J H(\mu_i, \nu_{\pi(\sigma^{-1}(i))}) = \\ &= \sum_{\sigma \in S_J} \sum_{\rho \in S_J} (-1)^\rho \prod_{i=1}^J H(\mu_i, \nu_{\rho(i)}) = J! \det(H(\mu_i, \nu_j))_{1 \leq i, j \leq J}. \end{aligned}$$

This completes the proof. \square

Now let $m \in \mathbb{Z}_+$, $0 < \xi_1 < \dots < \xi_m < 1$, $0 < \eta_1 < \dots < \eta_m < 1$, and let the alternation condition $\eta_{j+k-r} < \xi_j < \eta_{j+k}$ hold. For $t \in [0, 1]$, $\tau \in [0, 1]$ we set

$$G(t, \tau, \xi, \eta) = \frac{1}{(k-1)!(r-k-1)!m} \frac{K_{r,k}(t, \xi_1, \dots, \xi_m, \tau, \eta_1, \dots, \eta_m)}{K_{r,k}(\xi_1, \dots, \xi_m, \eta_1, \dots, \eta_m)}. \quad (25)$$

By Lemma 2 and the alternation condition, the denominator is strict positive. We denote

$$H(t, \tau) = \int_0^1 (t-s)_+^{k-1} (\tau-s)_+^{r-k-1} ds. \quad (26)$$

Then for a fixed $t \in (0, 1)$

$$H(t, \tau) \underset{r,k,t}{\asymp} 1 \quad \text{if } \tau \text{ is in a left neighborhood of } 1, \quad (27)$$

$$H(t, \tau) \underset{r,k,t}{\asymp} \tau^{r-k} \quad \text{if } \tau \text{ is in a right neighborhood of } 0. \quad (28)$$

Lemma 4. *The following assertions hold.*

1. For any $t \in [0, 1]$, $\tau \in [0, 1]$, $1 \leq j \leq m$ the equalities $G(\xi_j, \tau, \xi, \eta) = 0$ and $G(t, \eta_j, \xi, \eta) = 0$ hold; in particular,

$$\int_0^1 G(\xi_j, \tau, \xi, \eta) \varphi(\tau) d\tau = 0, \quad \varphi \in L_1^{\text{loc}}(0, 1), \quad (29)$$

$$\int_0^1 G(t, \eta_j, \xi, \eta) \varphi(\tau) d\tau = 0, \quad \varphi \in L_1^{\text{loc}}(0, 1). \quad (30)$$

2. If $x \in W_{p,g}^{r,k}[0, 1]$, then

$$x(t) - (-1)^{r-k} \int_0^1 G(t, \tau, \xi, \eta) x^{(r)}(\tau) d\tau = \sum_{j=1}^m c_j(x) H(t, \eta_j), \quad (31)$$

where $x \mapsto c_j(x)$ are linear continuous functionals on $\mathcal{W}_{p,g}^{r,k}[0, 1]$.

3. Let $(\bar{x}, \bar{y}, \bar{\theta}) \in SP_m$, let $0 < \xi_1 < \dots < \xi_m < 1$ be zeroes of \bar{x} , and let $0 < \eta_1 < \dots < \eta_m < 1$ be zeroes of \bar{y} . Then

$$\bar{x}(t) = (-1)^{r-k} \int_0^1 G(t, \tau, \xi, \eta) \bar{x}^{(r)}(\tau) d\tau, \quad \bar{y}(\tau) = (-1)^k \int_0^1 G(t, \tau, \xi, \eta) \bar{y}^{(r)}(t) dt. \quad (32)$$

In addition,

$$(-1)^{r-k} \overline{x}(t) G(t, \tau, \xi, \eta) \overline{x}^{(r)}(\tau) \geq 0. \quad (33)$$

Proof. From Lemma 3 and (21), (22), (25) it follows that

$$G(t, \tau, \xi, \eta) = \frac{1}{C(k-1)!(r-k-1)!} \begin{vmatrix} H(t, \tau) & H(\xi_1, \tau) & \dots & H(\xi_m, \tau) \\ H(t, \eta_1) & H(\xi_1, \eta_1) & \dots & H(\xi_m, \eta_1) \\ \dots & \dots & \dots & \dots \\ H(t, \eta_m) & H(\xi_1, \eta_m) & \dots & H(\xi_m, \eta_m) \end{vmatrix}, \quad (34)$$

$$C = \begin{vmatrix} H(\xi_1, \eta_1) & \dots & H(\xi_m, \eta_1) \\ \dots & \dots & \dots \\ H(\xi_1, \eta_m) & \dots & H(\xi_m, \eta_m) \end{vmatrix} > 0 \quad (35)$$

by Proposition 2 and Lemma 2. This implies the first assertion of Lemma.

Let us prove the second assertion. Indeed, from (26), (34) and (35) we get that

$$\begin{aligned} (-1)^{r-k} \int_0^1 G(t, \tau, \xi, \eta) x^{(r)}(\tau) d\tau &= \sum_{j=1}^m \tilde{c}_j(x) H(t, \eta_j) + \\ &+ \frac{(-1)^{r-k}}{(k-1)!(r-k-1)!} \int_{[0,1]} \left(\int_{[0,1]} (t-s)_+^{k-1} (\tau-s)_+^{r-k-1} ds \right) x^{(r)}(\tau) d\tau = \\ &= \sum_{j=1}^m \tilde{c}_j(x) H(t, \eta_j) + \\ &+ \frac{(-1)^{r-k}}{(k-1)!(r-k-1)!} \int_0^t (t-s)^{k-1} \int_s^1 (\tau-s)^{r-k-1} x^{(r)}(\tau) d\tau ds \stackrel{(6)}{=} \\ &= \sum_{j=1}^m \tilde{c}_j(x) H(t, \eta_j) + x(t), \end{aligned}$$

where $\tilde{c}_j(x)$ are linear functionals. We set $c_j(x) = -\tilde{c}_j(x)$. The values $c_j(x)$ are some linear combinations of the integrals

$$\int_0^1 H(\xi_i, \tau) x^{(r)}(\tau) d\tau = \int_0^1 H(\xi_i, \tau) g(\tau) \varphi(\tau) d\tau, \quad \varphi = \frac{x^{(r)}}{g}.$$

From (5), (27) and (28) it follows that the right part continuously depends on $\varphi \in L_p[0, 1]$. This implies the continuity of the functionals c_j on the space $\mathcal{W}_{p,g}^{r,k}[0, 1]$.

Let us prove the third assertion. By the second assertion, which is already proved,

$$\bar{x}(t) - (-1)^{r-k} \int_0^1 G(t, \tau, \xi, \eta) \bar{x}^{(r)}(\tau) d\tau = \sum_{j=1}^m c_j(\bar{x}) H(t, \eta_j). \quad (36)$$

By the condition of assertion 3 in Lemma, $\bar{x}(\xi_i) = 0$. This together with (29) implies that the left-hand side of (36) equals to zero in points ξ_i . Hence,

$$\sum_{j=1}^m c_j(\bar{x}) H(\xi_i, \eta_j) = 0, \quad 1 \leq i \leq m.$$

By (35), we have $c_j(\bar{x}) = 0$.

The second equality (32) can be proved similarly (here we apply (30)).

Let us prove (33). If $t < \xi_1 < \dots < \xi_m$, $\tau < \eta_1 < \dots < \eta_m$, then $G(t, \tau, \xi, \eta) \geq 0$ by Lemma 2 and (25). Let $t \in (\xi_i, \xi_{i+1})$, $\tau \in (\eta_j, \eta_{j+1})$. Then

$$G(t, \tau, \xi, \eta) \cdot (-1)^{i+j} \geq 0.$$

Since ξ_i are points of sign change of the function \bar{x} , and η_j are the points of sign change of the function $\bar{x}^{(r)}$ by the first equation (7), we get that the sign of $\bar{x}(t)G(t, \tau, \xi, \eta)\bar{x}^{(r)}(\tau)$ is constant. Therefore, it is sufficient to prove the inequality $(-1)^{r-k}\bar{x}(t)\bar{x}^{(r)}(\tau) \geq 0$ for t, τ from a small neighborhood of zero. Without loss of generality we may assume that $\bar{x}^{(r)}(\tau) \geq 0$ in a neighborhood of zero. Then $(-1)^m\bar{x}^{(r)}(\tau) \geq 0$ in a neighborhood of 1. Therefore, $(-1)^{m+r-k}\bar{x}^{(k)}(\tau) \geq 0$ in a neighborhood of 1, which implies that $(-1)^{r-k}\bar{x}^{(k)}(\tau) \geq 0$ in a neighborhood of 0. Hence, $(-1)^{r-k}\bar{x}(\tau) \geq 0$ in a neighborhood of 0. \square

The further arguments in obtaining the upper estimate are the same as in [5, §7], [30, p. 389–390].

3 Some properties of the kernel $G(t, \tau, \xi, \eta)$.

Let $0 < \xi_1 < \dots < \xi_n < 1$, $0 < \eta_1 < \dots < \eta_n < 1$, and let the alternation condition

$$\eta_{j+k-r} < \xi_j < \eta_{j+k} \quad (37)$$

hold. We set $\xi_0 := \eta_0 := 0$, $\xi_{n+1} := \eta_{n+1} := 1$, $\xi = (\xi_i)_{i=1}^n$, $\eta = (\eta_i)_{i=1}^n$. Let $\xi_{i-1} < t < \xi_i$, $\eta_{j-1} < \tau < \eta_j$. We set

$$\alpha_l = \xi_l, \quad 1 \leq l \leq i-1, \quad \alpha_i = t, \quad \alpha_l = \xi_{l-1}, \quad i+1 \leq l \leq n+1; \quad (38)$$

$$\beta_l = \eta_l, \quad 1 \leq l \leq j-1, \quad \beta_j = \tau, \quad \beta_l = \eta_{l-1}, \quad j+1 \leq l \leq n+1. \quad (39)$$

Then $\alpha_1 < \alpha_2 < \dots < \alpha_{n+1}$, $\beta_1 < \beta_2 < \dots < \beta_{n+1}$.

Applying Lemma 2 and (25), we obtain the following assertion.

Proposition 3. *If*

$$\beta_{l+k-r} < \alpha_l < \beta_{l+k} \quad (40)$$

for any l , then $G(t, \tau, \xi, \eta) \neq 0$.

For $1 \leq i \leq n+1$, $1 \leq j \leq n+1$ we set $\Delta_{i,j} = (\xi_{i-1}, \xi_i) \cap (\eta_{j-1}, \eta_j)$.

Proposition 4. *Let $\Delta_{i,j} \neq \emptyset$. Then*

$$i + k - r \leq j \leq i + k. \quad (41)$$

Proof. We have $1 \leq i \leq n+1$, $1 \leq j \leq n+1$.

If $\Delta_{i,j} \neq \emptyset$, then either $\xi_{i-1} \leq \eta_{j-1} < \xi_i$ or $\eta_{j-1} < \xi_{i-1} < \eta_j$.

Consider the first case. If $i = 1$, then $i + k - r < j$. If $i = n+1$, then $j < i + k$.

For $i \geq 2$ the inequality $\eta_{i-1+k-r} \stackrel{(37)}{<} \xi_{i-1}$ holds; hence, $\eta_{i-1+k-r} < \eta_{j-1}$ and $j > i + k - r$. For $i \leq n$ the inequality $\xi_i \stackrel{(37)}{<} \eta_{i+k}$ holds; therefore, $\eta_{j-1} < \xi_i < \eta_{i+k}$ and $j < i + k + 1$. Thus,

$$i + k - r < j \leq i + k. \quad (42)$$

Let us consider the second case. If $j = 1$, then $j < i + k$. If $j = n+1$, then $i - r + k < j$. For $j \geq 2$ we have $\xi_{j-1-k} \stackrel{(37)}{<} \eta_{j-1}$; therefore, $\xi_{j-1-k} < \eta_{j-1} < \xi_{i-1}$ and $i > j - k$. For $j \leq n$ the inequality $\eta_j \stackrel{(37)}{<} \xi_{j+r-k}$ holds; hence, $\xi_{i-1} < \eta_j < \xi_{j+r-k}$ and $j > i - 1 - r + k$. Thus,

$$i - r + k \leq j < i + k. \quad (43)$$

The union of inequalities (42) and (43) gives (41). \square

Lemma 5. *Let*

$$\Delta_{i,j} \neq \emptyset. \quad (44)$$

1. *If $i - r + k + 1 \leq j \leq i + k - 1$, then for any $\tau \in \Delta_{i,j}$, $t \in \Delta_{i,j}$ we have $G(t, \tau, \xi, \eta) \neq 0$.*
2. *If $j = i - r + k$, then for any $\tau \in \Delta_{i,j}$, $t \in \Delta_{i,j}$ such that $t > \tau$ we have $G(t, \tau, \xi, \eta) \neq 0$.*
3. *If $j = i + k$, then for any $\tau \in \Delta_{i,j}$, $t \in \Delta_{i,j}$ such that $t < \tau$, we have $G(t, \tau, \xi, \eta) \neq 0$.*

Proof. In all cases

$$t \in \Delta_{i,j}, \quad \tau \in \Delta_{i,j}; \quad (45)$$

therefore,

$$\xi_0 < \xi_1 < \cdots < \xi_{i-1} < t < \xi_i < \cdots < \xi_n < \xi_{n+1},$$

$$\eta_0 < \eta_1 < \cdots < \eta_{j-1} < \tau < \eta_j < \cdots < \eta_n < \eta_{n+1}.$$

We define the points α_l and β_l by formulas (38), (39). Let us prove that the alternation conditions $\beta_{l+k-r} < \alpha_l < \beta_{l+k}$ hold and apply Proposition 3.

First we prove that $\beta_{l+k-r} < \alpha_l$.

1. If $l \leq i-1$, then $\alpha_l \stackrel{(38)}{=} \xi_l$. Further, $l+k-r \leq i-1+k-r \stackrel{(41)}{\leq} j-1$. Hence, $\beta_{l+k-r} \stackrel{(39)}{=} \eta_{l+k-r}$. It remains to apply (37).

2. If $l = i$, then $\alpha_l \stackrel{(38)}{=} t$. By (41), $j \geq i+k-r$.

- If $i+k-r \leq j-1$, then $\beta_{i+k-r} \stackrel{(39)}{=} \eta_{i+k-r}$; hence, it is sufficient to check the inequality $\eta_{i+k-r} < t$. It follows from relations $t \stackrel{(45)}{>} \eta_{j-1} \geq \eta_{i+k-r}$.
- If $j = i-r+k$, then $\beta_{i+k-r} \stackrel{(39)}{=} \tau$. By conditions of Lemma (see assertion 2), we have $t > \tau$.

3. If $l \geq i+1$, then $\alpha_l \stackrel{(38)}{=} \xi_{l-1}$.

- Let $l+k-r \leq j-1$. Then $\beta_{l+k-r} \stackrel{(39)}{=} \eta_{l+k-r}$. The desired inequality follows from relations $\beta_{l+k-r} = \eta_{l+k-r} \leq \eta_{j-1} \stackrel{(44)}{<} \xi_i \leq \xi_{l-1} = \alpha_l$.
- Let $l+k-r = j$. Then $\beta_{l+k-r} \stackrel{(39)}{=} \tau$. The desired inequality follows from relations $\beta_{l+k-r} = \tau \stackrel{(45)}{<} \xi_i \leq \xi_{l-1} = \alpha_l$.
- Let $l+k-r \geq j+1$. Then $\beta_{l+k-r} \stackrel{(39)}{=} \eta_{l+k-r-1}$. It remains to apply (37).

Now we prove that $\alpha_l < \beta_{l+k}$.

1. If $l \leq i-1$, then $\alpha_l \stackrel{(38)}{=} \xi_l$.

- Let $l+k \leq j-1$. Then $\beta_{l+k} \stackrel{(39)}{=} \eta_{l+k}$. The desired inequality follows from (37).
- Let $l+k = j$. Then $\beta_{l+k} \stackrel{(39)}{=} \tau$. The desired inequality follows from relations $\alpha_l = \xi_l \leq \xi_{i-1} \stackrel{(45)}{<} \tau = \beta_{l+k}$.

- Let $l + k \geq j + 1$. Then $\beta_{l+k} \stackrel{(39)}{=} \eta_{l+k-1}$. The desired inequality follows from relations $\alpha_l = \xi_l \leq \xi_{i-1} \stackrel{(44)}{<} \eta_j \leq \eta_{l+k-1} = \beta_{l+k}$.
2. If $l = i$, then $\alpha_l \stackrel{(38)}{=} t$. By (41), $i + k \geq j$.
- Let $i + k > j$. Then $\beta_{i+k} \stackrel{(39)}{=} \eta_{i+k-1} \geq \eta_j \stackrel{(45)}{>} t = \alpha_i$.
 - Let $i + k = j$. Then $\beta_{i+k} \stackrel{(39)}{=} \tau$, and the inequality $t < \tau$ follows from the condition of Lemma (see assertion 3).
3. Let $l \geq i + 1$. Then $\alpha_l \stackrel{(38)}{=} \xi_{l-1}$. Since $l + k > i + k \stackrel{(41)}{\geq} j$, we have $\beta_{l+k} \stackrel{(39)}{=} \eta_{l+k-1}$. Hence, the desired inequality follows from (37).

This completes the proof. \square

Lemma 6. *Let $\Delta_{i,j} \neq \emptyset$, $\xi_{i-1} \neq \eta_{j-1}$, $\tau_* = \max\{\xi_{i-1}, \eta_{j-1}\}$. Then there exist $t_* \in (0, 1)$ and $\delta > 0$ such that $G(t, \tau, \xi, \eta) \neq 0$ for a.e. $(t, \tau) \in (t_* - \delta, t_* + \delta) \times (\tau_* - \delta, \tau_* + \delta)$.*

Proof. *Case $\xi_{i-1} > \eta_{j-1}$.* Then $\tau_* = \xi_{i-1}$ and $\xi_{i-1} < \eta_j$. We show that $i \geq j - k + 1$. Indeed, otherwise $i \stackrel{(41)}{=} j - k$, $\eta_{j-1} < \xi_{i-1} = \xi_{j-k-1}$; this contradicts with (37).

1. Let $i \geq j - k + 2$. We take $t_* \in \Delta_{i,j}$ and sufficiently small $\delta > 0$. Then $\tau_* < t_*$. If $\tau \in (\tau_*, \tau_* + \delta)$, $|t - t_*| < \delta$, then $\tau < t$, $t, \tau \in \Delta_{i,j}$, and by Lemma 5 we get $G(t, \tau, \xi, \eta) \neq 0$. Let $\tau \in (\tau_* - \delta, \tau_*)$. Since $\xi_{i-1} > \eta_{j-1}$, the points α_l and β_l are defined by formulas (38), (39), as in the case $\tau \in (\tau_*, \tau_* + \delta)$. Hence, it suffices to check that $\tau > \alpha_{j-k}$ and to apply Proposition 3. Indeed, since $j - k \leq i - 2$, we have $\alpha_{j-k} \stackrel{(38)}{=} \xi_{j-k} \leq \xi_{i-2} < \tau$.
2. Let $i = j - k + 1$. We take sufficiently small $\delta > 0$ and $t_* \in \Delta_{i-1,j}$. If $\tau \in (\tau_* - \delta, \tau_*)$, then for small $\delta > 0$ we have $\tau > t$. Since $i = j - k + 1$, the case $j = i - 1 - r + k$ is impossible. Hence, $G(t, \tau, \xi, \eta) \neq 0$ by Lemma 5 with $i := i - 1$, $j := j$. Let $\tau \in (\tau_*, \tau_* + \delta)$. Then points α_l, β_l are defined as in case $\tau \in (\tau_* - \delta, \tau_*)$: $\alpha_l = \xi_l$ for $l \leq i - 2$, $\alpha_{i-1} = t$, $\alpha_l = \xi_{l-1}$ for $l \geq i$, β_l is defined by (39). Thus, in order to check the conditions (40) it is sufficient to prove that $\tau < \alpha_{j+r-k}$. Indeed, $j + r - k = i + r - 1 \geq i + 1$; therefore, $\alpha_{j+r-k} = \xi_{j+r-k-1} \geq \xi_i > \tau$.

Case $\xi_{i-1} < \eta_{j-1}$. Then $i \leq j + r - k - 1$ (otherwise, $i \stackrel{(41)}{=} j + r - k$, $\xi_{i-1} < \eta_{j-1} = \eta_{i-r+k-1}$, which contradicts with (37)). Moreover,

$$\xi_i > \eta_{j-1}, \quad \tau_* = \eta_{j-1}. \quad (46)$$

1. Let $i \leq j + r - k - 2$. We take $t_* \in \Delta_{i,j-1}$. Then $\tau_* = \eta_{j-1} > t_*$. If $|t - t_*| < \delta$, $\tau \in (\tau_* - \delta, \tau_*)$, then for small $\delta > 0$ the inequality $t < \tau$ holds and $\tau \in \Delta_{i,j-1}$. Here α_l are defined by (38), $\beta_l = \eta_l$ for $l \leq j - 2$, $\beta_{j-1} = \tau$, $\beta_l = \eta_{l-1}$ for $l \geq j$. By Lemma 5 with $i := i$, $j := j - 1$, $G(t, \tau, \xi, \eta) \neq 0$. Let $\tau \in (\tau_*, \tau_* + \delta)$. Then the numbers α_l are defined by (38) as in the case $\tau \in (\tau_* - \delta, \tau_*)$. The numbers β_l are defined as follows: $\beta_l = \eta_l$ for $l \leq j - 1$, $\beta_j = \tau$, $\beta_l = \eta_{l-1}$ for $l \geq i + 1$. Thus, in definition of β_l we change β_{j-1} and β_j : $\beta_{j-1} = \eta_{j-1}$, $\beta_j = \tau$. Hence, in order to prove (40) it is sufficient to show that

$$\alpha_{j-1-k} < \eta_{j-1} < \alpha_{j-1+r-k}, \quad (47)$$

$$\alpha_{j-k} < \tau < \alpha_{j+r-k}. \quad (48)$$

Since $t_* \in \Delta_{i,j-1}$, we have $\xi_{i-1} < t < \xi_i$. By conditions of Lemma, $\Delta_{i,j} \neq \emptyset$. Therefore, by (41), $j - 1 - k \leq i - 1$. Hence, $\alpha_{j-1-k} \stackrel{(38)}{=} \xi_{j-1-k} < \eta_{j-1}$ by (37). Further, since $i \leq j + r - k - 2$, we have $j - 1 + r - k \geq i + 1$ and $\alpha_{j-1+r-k} \stackrel{(38)}{=} \xi_{j-2+r-k} \geq \xi_i \stackrel{(46)}{>} \eta_{j-1}$. This completes the proof of (47). Let us check (48). We have $\alpha_{j+r-k} = \xi_{j+r-k-1} \geq \xi_{i+1} > \tau$. In order to prove that $\tau > \alpha_{j-k}$ it is sufficient to show that $\eta_{j-1} > \alpha_{j-k}$ (see the second inequality of (46)). If $j - k \leq i - 1$, then $\alpha_{j-k} \stackrel{(38)}{=} \xi_{j-k} \leq \xi_{i-1} < \eta_{j-1}$. If $j - k = i$, then $\alpha_{j-k} = \alpha_i \stackrel{(38)}{=} t < \eta_{j-1}$ (since $t_* \in \Delta_{i,j-1}$ and δ is sufficiently small).

2. Let $i = j + r - k - 1$. We take $t_* \in \Delta_{i,j}$. Then $t_* > \tau_*$. If $|t - t_*| < \delta$, $\tau \in (\tau_*, \tau_* + \delta)$, then $\tau < t$ for small $\delta > 0$. The case $i + k = j$ is impossible, since $r \geq 2$. Therefore, $G(t, \tau, \xi, \eta) \neq 0$ by Lemma 5. Let $\tau \in (\tau_* - \delta, \tau_*)$. Then $\beta_{j-1} = \tau$, $\beta_j = \eta_{j-1}$, other numbers α_l and β_l are the same as for $\tau \in (\tau_*, \tau_* + \delta)$; i.e., they are defined by (38) and (39). Hence, in order to check the condition (40), it is sufficient to prove that

$$\alpha_{j-k-1} < \tau < \alpha_{j-1+r-k}, \quad (49)$$

$$\alpha_{j-k} < \eta_{j-1} < \alpha_{j+r-k}. \quad (50)$$

We have $j - k - 1 = i - r \leq i - 1$; therefore, $\alpha_{j-k-1} \stackrel{(38)}{=} \xi_{j-k-1} \leq \xi_{i-1} < \tau$ (the last inequality holds since $\xi_{i-1} < \eta_{j-1}$, and τ is close to $\tau_* \stackrel{(46)}{=} \eta_{j-1}$). Further, $j - 1 + r - k = i$; hence, $\alpha_{j-1+r-k} = \alpha_i \stackrel{(38)}{=} t > \tau$. This completes the proof of (49). Let us check (50). We have $\alpha_{j+r-k} = \alpha_{i+1} \stackrel{(38)}{=} \xi_i \stackrel{(46)}{>} \eta_{j-1}$. Since $r \geq 2$, we get $j - k = i - r + 1 \leq i - 1$, and $\alpha_{j-k} \stackrel{(38)}{=} \xi_{j-k} \leq \xi_{i-1} < \eta_{j-1}$.

This completes the proof. \square

Lemma 7. Let $r \geq 3$, $\Delta_{i,j} \neq \emptyset$, $\tau_* = \xi_{i-1} = \eta_{j-1}$. Then there exist $\delta > 0$ and $t_* \in (0, 1)$ such that $G(t, \tau, \xi, \eta) \neq 0$ for a.e. $(t, \tau) \in (t_* - \delta, t_* + \delta) \times (\tau_* - \delta, \tau_* + \delta)$.

Proof. First we prove that $i - 1 + k \neq j$ or $i \neq j - 1 + r - k$. Indeed, otherwise $i = j - k + 1$, $i = j - k - 1 + r$; i.e., $1 = r - 1$. It contradicts with condition $r \geq 3$.

We show that

$$i + k \neq j, \quad i - r + k \neq j. \quad (51)$$

Indeed, if $i + k = j$ or $i = j + r - k$, then $\xi_{i-1} = \eta_{i+k-1}$ or $\xi_{i-1} = \eta_{i-r+k-1}$. This contradicts with the alternation condition (37).

Case $i \neq j + 1 - k$. We take $t_* \in \Delta_{i,j}$.

Let $\tau \in (\tau_*, \tau_* + \delta)$, $|t - t_*| < \delta$. Then $t \in \Delta_{i,j}$, $\tau \in \Delta_{i,j}$ for small $\delta > 0$. Since $i + k \neq j$, $i - r + k \neq j$, we get $G(t, \tau, \xi, \eta) \neq 0$ by Lemma 5.

Let $\tau \in (\tau_* - \delta, \tau_*)$. Then

$$\alpha_l = \xi_l, \quad l \leq i - 1, \quad \alpha_i = t, \quad \alpha_l = \xi_{l-1}, \quad l \geq i + 1, \quad (52)$$

$$\beta_l = \eta_l, \quad l \leq j - 2, \quad \beta_{j-1} = \tau, \quad \beta_l = \eta_{l-1}, \quad l \geq j. \quad (53)$$

Let us prove that

$$\beta_{l+k-r} < \alpha_l. \quad (54)$$

1. If $l \leq i - 1$, then $\alpha_l \stackrel{(52)}{=} \xi_l$, $l + k - r \leq i - 1 + k - r \stackrel{(41)}{\leq} j - 1$, and $\beta_{l+k-r} \stackrel{(53)}{\leq} \tau < \xi_{i-1} = \xi_l$.
2. If $l = i$, then $\alpha_l \stackrel{(52)}{=} t$. Further, $l + k - r = i + k - r \stackrel{(41)}{\leq} j$; hence, $\beta_{l+k-r} \leq \beta_j \stackrel{(53)}{=} \eta_{j-1}$. It remains to apply the inequality $t > \eta_{j-1}$ (it holds since $t_* \in \Delta_{i,j}$ and δ is sufficiently small).
3. If $l \geq i + 1$, then $\alpha_l \stackrel{(52)}{=} \xi_{l-1}$. Let $\beta_{l+k-r} \leq \tau$. Then (54) follows from inequalities $\tau < \xi_i \leq \xi_{l-1}$. If $\beta_{l+k-r} > \tau$, then $\beta_{l+k-r} \stackrel{(53)}{=} \eta_{l+k-r-1}$, and (54) follows from (37).

Now we prove that

$$\alpha_l < \beta_{l+k}. \quad (55)$$

1. If $l \leq i - 1$, then $\alpha_l \stackrel{(52)}{=} \xi_l$.
 - If $l + k \leq j - 2$, then $\beta_{l+k} \stackrel{(53)}{=} \eta_{l+k}$, and (55) follows from (37).

- If $l + k = j - 1$, then $\beta_{l+k} \stackrel{(53)}{=} \tau > \xi_{i-2}$ (the last inequality holds for small $\delta > 0$, since $\tau_* = \xi_{i-1}$ and $\tau > \tau_* - \delta$). Hence, if $l + k = j - 1$ and $l \leq i - 2$, then $\beta_{l+k} > \xi_{i-2} \geq \xi_l = \alpha_l$. Let $l + k = j - 1$, $l = i - 1$. Then $\xi_l = \xi_{i-1} = \eta_{j-1} = \eta_{l+k}$; this contradicts with (37).
 - If $l + k \geq j$, then $\beta_{l+k} = \eta_{l+k-1}$. Hence, (55) is equivalent to $\xi_l < \eta_{l+k-1}$. We have $\xi_l \leq \xi_{i-1} = \eta_{j-1} \leq \eta_{l+k-1}$; moreover, $\xi_l = \eta_{l+k-1}$ holds only for $l = i - 1$, $l + k = j$. Then $i = j + 1 - k$, which contradicts with our assumption.
2. If $l = i$, then $\alpha_l \stackrel{(52)}{=} t \in (\eta_{j-1}, \eta_j)$. Further, $l + k = i + k \stackrel{(41)}{\geq} j$; thus, $\beta_{l+k} \stackrel{(53)}{=} \eta_{i+k-1}$. By (41) and (51), we have $i + k - 1 \geq j$; therefore, $\beta_{l+k} \geq \eta_j > t = \alpha_l$.
3. If $l \geq i + 1$, then $\alpha_l \stackrel{(52)}{=} \xi_{l-1}$, $l + k \geq i + 1 + k \stackrel{(41)}{>} j$; hence, $\beta_{l+k} \stackrel{(53)}{=} \eta_{l+k-1}$, and (55) follows from (37).

From (54), (55) and Proposition 3 it follows that $G(t, \tau, \xi, \eta) \neq 0$.

Case $i \neq j - 1 + r - k$. We take $t_* \in \Delta_{i-1, j-1}$.

Let $\tau \in (\tau_* - \delta, \tau_*)$, $|t - t_*| < \delta$. Then $t \in \Delta_{i-1, j-1}$, $\tau \in \Delta_{i-1, j-1}$. Since $i - 1 + k \neq j - 1$, $i - 1 - r + k \neq j - 1$ by (51), we have $G(t, \tau, \xi, \eta) \neq 0$ by Lemma 5.

Let now $\tau \in (\tau_*, \tau_* + \delta)$, $\delta > 0$ is sufficiently small. Then

$$\alpha_l = \xi_l, \quad l \leq i - 2, \quad \alpha_{i-1} = t, \quad \alpha_l = \xi_{l-1}, \quad l \geq i, \quad (56)$$

$$\beta_l = \eta_l, \quad l \leq j - 1, \quad \beta_j = \tau, \quad \beta_l = \eta_{l-1}, \quad l \geq j + 1. \quad (57)$$

We prove that

$$\alpha_l < \beta_{l+k}. \quad (58)$$

1. If $l \leq i - 2$, then $\alpha_l \stackrel{(56)}{=} \xi_l$. In the case $l + k \leq j - 1$ we get $\beta_{l+k} \stackrel{(57)}{=} \eta_{l+k}$, and (58) follows from (37). If $l + k = j$, then $\beta_{l+k} \stackrel{(57)}{=} \tau > \xi_{i-1} > \xi_l$. If $l + k \geq j + 1$, then $\beta_{l+k} \stackrel{(57)}{=} \eta_{l+k-1} \geq \eta_j > t > \xi_{i-2} \geq \xi_l$.
2. If $l = i - 1$, then $\alpha_l \stackrel{(56)}{=} t$, $\beta_{l+k} = \beta_{i-1+k}$. By (41), $j \leq i + k$. If $j \leq i - 1 + k$, then $\beta_{i-1+k} \geq \beta_j \stackrel{(57)}{=} \tau > t$. Let $j = i + k$. Then $\beta_{i-1+k} = \beta_{j-1} \stackrel{(57)}{=} \eta_{j-1} > t$.
3. If $l \geq i$, then $\alpha_l \stackrel{(56)}{=} \xi_{l-1}$. We have $l + k \geq i + k \stackrel{(41)}{\geq} j$. Let $l = i$. Then $\alpha_l = \xi_{i-1} < \tau \stackrel{(57)}{=} \beta_j \leq \beta_{l+k}$. If $l \geq i + 1$, then $\alpha_l = \xi_{l-1} \stackrel{(37)}{<} \eta_{l+k-1} \stackrel{(57)}{=} \beta_{l+k}$ (the last equality holds since $l + k \geq i + 1 + k \stackrel{(41)}{\geq} j + 1$).

We prove that

$$\beta_{l+k-r} < \alpha_l. \quad (59)$$

1. If $l \leq i - 2$, then $\alpha_l \stackrel{(56)}{=} \xi_l$. We show that $l + k - r \leq j - 1$. Indeed, by (41), $i \leq j + r - k$; hence, $l + k - r \leq i - 2 + k - r \leq j - 1$. Therefore, $\beta_{l+k-r} \stackrel{(57)}{=} \eta_{l+k-r}$, and (59) follows from the alternation condition for ξ, η .
2. If $l = i - 1$, then $\alpha_l \stackrel{(56)}{=} t > \eta_{j-2}$. Therefore, it is sufficient to prove that $\beta_{l+k-r} \leq \eta_{j-2}$. By (41) and (51), $i \leq j + r - k - 1$; hence, $l + k - r = i - 1 + k - r \leq j - 2$. Therefore, $\beta_{l+k-r} \stackrel{(57)}{=} \eta_{l+k-r} \leq \eta_{j-2}$.
3. If $l \geq i$, then $\alpha_l \stackrel{(56)}{=} \xi_{l-1}$.
 - Let $l + k - r \geq j + 1$. Then $\beta_{l+k-r} \stackrel{(57)}{=} \eta_{l+k-r-1}$, and (59) follows from the alternation condition for ξ, η .
 - If $l + k - r = j$, then $\beta_{l+k-r} \stackrel{(57)}{=} \tau < \xi_i$. Hence, it is sufficient to check that $\xi_{l-1} \geq \xi_i$, which means that $l - 1 \geq i$. Suppose the contrary: let $l = i$. Then $i = j + r - k$, which contradicts with (51).
 - Let $l + k - r \leq j - 1$. Then $\beta_{l+k-r} \stackrel{(57)}{=} \eta_{l+k-r}$, and (59) is equivalent to the inequality $\xi_{l-1} > \eta_{l+k-r}$. We have $\xi_{l-1} \geq \xi_{i-1} = \eta_{j-1} \geq \eta_{l+k-r}$; in addition, $\xi_{l-1} = \eta_{l+k-r}$ only if $l = i$, $l + k - r = j - 1$. Then $i = j - 1 + r - k$, which contradicts with our assumption.

From (58), (59) and Proposition 3 it follows that $G(t, \tau, \xi, \eta) \neq 0$. This completes the proof. \square

Corollary 1. *Let $\tau_* \in (0, 1)$. Suppose that $r \geq 3$, or $r = 2$ and $\tau_* \notin \{\eta_j : 1 \leq j \leq n, \eta_j = \xi_j\}$. Then there exist $\delta > 0$ and $t_* \in (0, 1)$ such that $G(t, \tau, \xi, \eta) \neq 0$ for a.e. $(t, \tau) \in (t_* - \delta, t_* + \delta) \times (\tau_* - \delta, \tau_* + \delta)$.*

4 Auxiliary assertions for $r = 2$

In this section we obtain analogues of Lemmas 10 and 11 from [30]. The main part of the proof is similar to arguments from [30, §5].

Since $r = 2$, $1 \leq k \leq r - 1$, we have $k = 1$. From (26) it follows that

$$H(t, \tau) = \min\{t, \tau\}. \quad (60)$$

Since the operator $\tilde{I}_{r,g,v}^k : L_p[0, 1] \rightarrow L_q[0, 1]$ is bounded, from (5) it follows that

$$\int_0^T v^q(t) t^q dt < \infty \quad \text{for any } 0 < T < 1. \quad (61)$$

Let $(\bar{x}, \bar{y}, \bar{\theta}) \in SP_n$. Then (see (7))

$$\begin{cases} \ddot{\bar{x}} = -g^{p'} \bar{y}_{(p')}, & \ddot{\bar{y}} = -\bar{\theta}^q v^q \bar{x}_{(q)}, \\ \bar{x}(0) = \dot{\bar{x}}(1) = \bar{y}(0) = \dot{\bar{y}}(1) = 0, \\ \left\| \frac{\ddot{\bar{x}}}{g} \right\|_{L_p[0,1]} = 1, \end{cases} \quad (62)$$

$0 < \xi_1 < \xi_2 < \dots < \xi_n < 1$ are points of sign change of the function \bar{x} , $0 < \eta_1 < \eta_2 < \dots < \eta_n < 1$ are points of sign change of the function \bar{y} . From the Rolle's theorem and Proposition 1 it follows that $\ddot{\bar{x}}$ has exactly n points of sign change. Without loss of generality we may assume that $\bar{x}(t) > 0$ in the right punctured neighborhood of zero. Hence, $\ddot{\bar{x}}(t) > 0$ in the right punctured neighborhood of zero. Then

1. if n is even, then $\dot{\bar{x}}(t) > 0$ and $\ddot{\bar{x}}(t) \leq 0$ in the left punctured neighborhood of 1. By the first equation (62), $\bar{y}(t) > 0$ in the left punctured neighborhood of 1. Since n is even, $\bar{y}(t) > 0$ in the right punctured neighborhood of zero.
2. if n is odd, then $\dot{\bar{x}}(t) < 0$ and $\ddot{\bar{x}}(t) \geq 0$ in the left punctured neighborhood of 1. By the first equation (62), $\bar{y}(t) < 0$ in the left punctured neighborhood of 1. Since n is, $\bar{y}(t) > 0$ in the right punctured neighborhood of zero.

Thus,

$$\bar{x}(t) > 0, \quad \bar{y}(t) > 0 \quad \text{in the right neighborhood of zero.} \quad (63)$$

Denote $\xi_0 = \eta_0 = 0$, $\xi_{n+1} = \eta_{n+1} = 1$.

Let

$$\Lambda := \{l = \overline{1, n} : \eta_i = \xi_i\} \neq \emptyset. \quad (64)$$

We denote the elements of Λ by l_s , $1 \leq s \leq m$. Also we set $l_0 = 0$, $l_{m+1} = n + 1$.

For $f \in W_{p,g}^{2,1}[0, 1]$ we denote

$$Q_L f(t) = \int_0^1 G(t, \tau, \xi, \eta) \ddot{f}(\tau) d\tau. \quad (65)$$

Then

$$Q_L f(\xi_j) \stackrel{(29)}{=} 0, \quad 1 \leq j \leq n, \quad (66)$$

$$P_L f := f + Q_L f \stackrel{(31),(65)}{\in} L_n := \left\{ \sum_{j=1}^n c_j h_j : c_j \in \mathbb{R}, \quad 1 \leq j \leq n \right\}, \quad (67)$$

where

$$h_j(t) = H(t, \eta_j) \stackrel{(60)}{=} \min\{t, \eta_j\}. \quad (68)$$

Let $\psi_1(t) = \min\{t, \eta_1\}$. For $2 \leq j \leq n$ we set

$$\psi_j(t) = \begin{cases} 0, & 0 \leq t \leq \eta_{j-1}, \\ t - \eta_{j-1}, & \eta_{j-1} \leq t \leq \eta_j, \\ \eta_j - \eta_{j-1}, & \eta_j \leq t \leq 1. \end{cases} \quad (69)$$

Then

$$h_j = \sum_{i=1}^j \psi_i; \quad (70)$$

hence,

$$L_n = \text{span} \{ \psi_j \}_{j=1}^n. \quad (71)$$

For $0 \leq s \leq m-1$ we denote

$$\eta^{(s)} = (\eta_{l_s+1}, \eta_{l_s+2}, \dots, \eta_{l_{s+1}}), \quad \xi^{(s)} = (\xi_{l_s+1}, \xi_{l_s+2}, \dots, \xi_{l_{s+1}}).$$

In addition, we set $\eta^{(m)} = (\eta_{l_m+1}, \eta_{l_m+2}, \dots, \eta_n)$, $\xi^{(m)} = (\xi_{l_m+1}, \xi_{l_m+2}, \dots, \xi_n)$. For $f \in W_{p,g}^{2,1}[0, 1]$, $0 \leq s \leq m$ we set

$$Q_s^0 f(t) = \int_{\eta_s}^{\eta_{s+1}} G(t, \tau, \xi^{(s)}, \eta^{(s)}) \ddot{f}(\tau) d\tau.$$

The following assertion is the analogue of Lemma 5 from [30].

Lemma 8. *The equality $Q_L f|_{[\eta_s, \eta_{s+1}]} = Q_s^0 f$ holds.*

Proof. Let $0 \leq s \leq m-1$. By direct calculations it can be checked that

$$\int_{\eta_s}^{\eta_{s+1}} \min\{t - \eta_s, \tau - \eta_s\} \ddot{f}(\tau) d\tau = -f(t) + f(\eta_s) + (t - \eta_s) \dot{f}(\eta_{s+1}). \quad (72)$$

In addition,

$$(-f(t) + f(\eta_s) + (t - \eta_s) \dot{f}(\eta_{s+1}))|_{t=\eta_s} = 0,$$

$$\left. \frac{d}{dt}(-f(t) + f(\eta_s) + (t - \eta_s)\dot{f}(\eta_{s+1})) \right|_{t=\eta_{s+1}} = 0.$$

Therefore,

$$\int_{\eta_s}^{\eta_{s+1}} \min\{t - \eta_s, \tau - \eta_s\} \ddot{f}(\tau) d\tau - Q_s^0 f(t) \in \text{span}\{\psi_j\}_{j=l_s+1}^{l_{s+1}} \quad (73)$$

(it is the analogue of formulas (67) and (71)). Since for any $t \in [\eta_s, \eta_{s+1}]$ we have

$t - \eta_s = \sum_{j=l_s+1}^{l_{s+1}} \psi_j(t)$, by (72) and (73) there are numbers $\{b_j\}_{j=l_s+1}^{l_{s+1}}$ such that

$$f(t) - f(\eta_s) + Q_s^0 f(t) = \sum_{j=l_s+1}^{l_{s+1}} b_j \psi_j(t), \quad \eta_s \leq t \leq \eta_{s+1}. \quad (74)$$

By (67) and (70), there are numbers $\{c_j\}_{j=1}^n$ such that $f(t) + Q_L f(t) = \sum_{j=1}^n c_j \psi_j$.

For $j \geq l_{s+1} + 1$ we have $\psi_j|_{[\eta_s, \eta_{s+1}]} \stackrel{(69)}{=} 0$, for $j \leq l_s$ we have $\psi_j|_{[\eta_s, \eta_{s+1}]} \equiv \text{const}$.

Hence, there is $A \in \mathbb{R}$ such that $f(t) + Q_L f(t) = A + \sum_{j=l_s+1}^{l_{s+1}} c_j \psi_j$. Since $Q_L f(\eta_s) \stackrel{(64)}{=}$

$Q_L f(\xi_{l_s}) \stackrel{(66)}{=} 0$ and $\psi_j(\eta_s) \stackrel{(69)}{=} 0$, $l_s + 1 \leq j \leq l_{s+1}$, we get $A = f(\eta_s)$ and

$$f(t) - f(\eta_s) + Q_L f(t) = \sum_{j=l_s+1}^{l_{s+1}} c_j \psi_j(t), \quad \eta_s \leq t \leq \eta_{s+1}. \quad (75)$$

Since $Q_L f(\xi_i) \stackrel{(66)}{=} 0$ and similarly $Q_s^0 f(\xi_i) = 0$, $l_s + 1 \leq i \leq l_{s+1}$, we get from (74) and (75) that

$$\sum_{j=l_s+1}^{l_{s+1}} (c_j - b_j) \psi_j(\xi_i) = 0, \quad l_s + 1 \leq i \leq l_{s+1}.$$

Let us prove that the matrix $(\psi_j(\xi_i))_{l_s+1 \leq i, j \leq l_{s+1}}$ is non-degenerate. By (68) and (70), it holds if and only if the matrix $(\min(\xi_i, \eta_j))_{l_s+1 \leq i, j \leq l_{s+1}}$ is non-degenerate; the last property follows from (60) and analogue of (35) for $l_s + 1 \leq i, j \leq l_{s+1}$ (see Proposition 2 and Lemma 2).

Thus, $c_j = b_j$. This together with (74) and (75) yields the assertion of Lemma for $s < m$.

Let $s = m$. Then from (72), the condition $\dot{f}(1) = 0$ and the equality

$$\int_{\eta_m}^1 \min\{t - \eta_m, \tau - \eta_m\} \ddot{f}(\tau) d\tau - Q_m^0 f(t) \in \text{span}\{\psi_j\}_{j=l_m+1}^n$$

it follows that

$$f(t) - f(\eta_{l_m}) + Q_m^0 f(t) = \sum_{j=l_m+1}^n b_j \psi_j(t), \quad \eta_{l_m} \leq t \leq 1.$$

Similarly as formula (75) it can be proved that

$$f(t) - f(\eta_{l_m}) + Q_L f(t) = \sum_{j=l_m+1}^n c_j \psi_j(t), \quad \eta_{l_m} \leq t \leq 1.$$

After that we argue similarly as for $s < m$. □

For $0 \leq \alpha < \beta \leq 1$ we set $\Phi_{[\alpha, \beta]}(f) = \int_{\alpha}^{\beta} |vf|^q dt$. Denote

$$\Phi_s(f) = \begin{cases} \Phi_{[\eta_{s-1}, \eta_s]}(f), & 1 \leq s \leq m, \\ \Phi_{[\eta_m, \eta_{m+1}]}(f), & s = m+1, \end{cases}$$

$$\varphi_s = \psi_{l_s}, \quad 1 \leq s \leq m, \tag{76}$$

$$\varphi_{m+1} = \eta_{l_{m+1}} - \eta_{l_m} - \psi_{l_{m+1}} \quad \text{for } l_m < n, \quad \varphi_{m+1} = 1 \quad \text{for } l_m = n. \tag{77}$$

The following assertion can be proved by direct calculations (see, e.g., [12], p. 268, Exercise 8.29).

Proposition 5. *Let μ be a measure, $1 < q < \infty$, $\Phi(f) = \int_0^1 |f(t)|^q d\mu(t)$. Then the function Φ has the Lagrange variation, which is equal to*

$$\Phi'(f)[h] = q \int_0^1 (f)_{(q)} h dt. \tag{78}$$

Remark 1. *This function is Fréchet differentiable and even in a more strong sense; it follows from the uniform smoothness of the space L_q for $1 < q < \infty$ (see [12], §9).*

The following lemma is similar to Proposition 6 from [30].

Lemma 9. *The following inequalities hold:*

$$\dot{\bar{x}}(\eta_s) \Phi'_s(\bar{x})[\varphi_s] < 0, \quad 1 \leq s \leq m, \quad \dot{\bar{x}}(\eta_m) \Phi'_{m+1}(\bar{x})[\varphi_{m+1}] > 0.$$

Proof. We first consider the case $1 \leq s \leq m$. We have

$$\Phi'_s(\bar{x})[\varphi_s] \stackrel{(69),(76),(78)}{=} q \int_{\eta_{s-1}}^{\eta_s} v^q(t) \bar{x}_{(q)}(t) (t - \eta_{s-1}) dt \stackrel{(62)}{=} -q \bar{\theta}^{-q} \int_{\eta_{s-1}}^{\eta_s} \ddot{\bar{y}}(t) (t - \eta_{s-1}) dt =: I_s. \quad (79)$$

If $l_s > 1$, then $\dot{\bar{y}} \in AC[\eta_{s-1}, \eta_s]$; we apply the formula of integration by parts and obtain

$$\int_{\eta_{s-1}}^{\eta_s} \ddot{\bar{y}}(t) (t - \eta_{s-1}) dt = (t - \eta_{s-1}) \dot{\bar{y}}(t) \Big|_{\eta_{s-1}}^{\eta_s} - \int_{\eta_{s-1}}^{\eta_s} \dot{\bar{y}}(t) dt = (\eta_s - \eta_{s-1}) \dot{\bar{y}}(\eta_s)$$

(here we apply the equalities $\bar{y}(\eta_j) = 0$, $1 \leq j \leq n$). Hence, there exists $A_s > 0$ such that

$$I_s = -A_s \dot{\bar{y}}(\eta_s), \quad 1 \leq s \leq m, \quad l_s > 1. \quad (80)$$

Let $s = 1$ and $l_1 = 1$. Then $\eta_{l_1-1} = 0$. Since $\bar{x} \in L_{q,v}[0, 1]$, we have $v^{q-1} \bar{x}_{(q)} \in L_{q'}[0, 1]$. This together with (61) and the Hölder inequality yields that the function $v^q(t) \bar{x}_{(q)}(t)t$ is Lebesgue integrable. By (62), the function $\ddot{\bar{y}}(t)t$ is also Lebesgue integrable. From the formula of integration by parts we get for small $\varepsilon > 0$

$$\int_{\varepsilon}^{\eta_1} \ddot{\bar{y}}(t)t dt = t \dot{\bar{y}}(t) \Big|_{\varepsilon}^{\eta_1} - \int_{\varepsilon}^{\eta_1} \dot{\bar{y}}(t) dt = -\varepsilon \dot{\bar{y}}(\varepsilon) + \eta_1 \dot{\bar{y}}(\eta_1) - \bar{y}(\eta_1) + \bar{y}(\varepsilon).$$

The both parts of this equality have a limit as $\varepsilon \rightarrow 0+$; here $\bar{y}(\eta_1) = 0$, $\bar{y}(\varepsilon) \xrightarrow{\varepsilon \rightarrow 0+} 0$ by (62). Let $\varepsilon \dot{\bar{y}}(\varepsilon) \xrightarrow{\varepsilon \rightarrow 0+} c$. If $c \neq 0$, then for small $\delta > 0$

$$\int_0^{\delta} |\dot{\bar{y}}| dt \geq \frac{|c|}{2} \int_0^{\delta} \frac{dt}{t}.$$

Since \bar{y} is absolutely continuous on $[0, 1]$, the left-hand side has a finite limit. The right-hand side is infinite, which leads to a contradiction. Hence, $c = 0$ and $\int_{\varepsilon}^{\eta_1} \ddot{\bar{y}}(t)t dt \rightarrow \eta_1 \dot{\bar{y}}(\eta_1)$ as $\varepsilon \rightarrow +0$. Thus, there exists $A_1 > 0$ such that

$$I_1 = -A_1 \dot{\bar{y}}(\eta_{l_1}), \quad l_1 = 1. \quad (81)$$

Let now $s = m + 1$. If $l_m < n$, then

$$\Phi'_{m+1}(\bar{x})[\varphi_{m+1}] \stackrel{(69),(77),(78)}{=} q \int_{\eta_m}^{\eta_{m+1}} v^q(t) \bar{x}_{(q)}(t) (\eta_{m+1} - t) dt \stackrel{(62)}{=}$$

$$= -q\bar{\theta}^{-q} \int_{\eta_m}^{\eta_{m+1}} \ddot{y}(t)(\eta_{m+1} - t) dt =: I_{m+1}.$$

Similarly as (80) we prove that

$$I_{m+1} = A_{m+1} \dot{\bar{y}}(\eta_m). \quad (82)$$

Let $l_m = n$. Then

$$\Phi'_{m+1}(\bar{x})[\varphi_{m+1}] \stackrel{(77),(78)}{=} q \int_{\eta_n}^1 v^q(t) \bar{x}_{(q)}(t) dt \stackrel{(62)}{=} -q\bar{\theta}^{-q} \int_{\eta_n}^1 \ddot{y}(t) dt =: I_{m+1}.$$

Since $\dot{\bar{y}}$ is absolutely continuous in a left neighborhood of 1 and $\dot{\bar{y}}(1) = 0$, then

$$I_{m+1} = q\bar{\theta}^{-q} \dot{\bar{y}}(\eta_n). \quad (83)$$

From (80), (81), (82), (83) it follows that in order to complete the proof it is sufficient to check that $\dot{\bar{x}}(\eta_{l_s}) \dot{\bar{y}}(\eta_{l_s}) > 0$, $1 \leq s \leq m$ (recall that by assertion 3 of Proposition 1 we have $\dot{\bar{x}}(\eta_{l_s}) \dot{\bar{y}}(\eta_{l_s}) \neq 0$).

Let l_s be even. By (63), in the left neighborhood of η_{l_s} we have $\bar{x}(t) < 0$, $\bar{y}(t) < 0$, and in the right neighborhood of η_{l_s} we have $\bar{x}(t) > 0$, $\bar{y}(t) > 0$. Hence, $\dot{\bar{x}}(\eta_{l_s}) > 0$, $\dot{\bar{y}}(\eta_{l_s}) > 0$. Similarly for odd l_s we get $\dot{\bar{x}}(\eta_{l_s}) < 0$, $\dot{\bar{y}}(\eta_{l_s}) < 0$. \square

Now we introduce notation from [30].

For $\mathbf{c} = (c_0, c_1, \dots, c_m) \in \mathbb{R}^{m+1}$ we denote

$$\psi_{\mathbf{c}} = \sum_{s=0}^m c_s \frac{\ddot{\bar{x}}}{g} \cdot \chi_{[\eta_s, \eta_{s+1}]}. \quad (84)$$

Let

$$\mathcal{C} = \left\{ \mathbf{c} \in \mathbb{R}^{m+1} : \|\psi_{\mathbf{c}}\|_{L_p[0,1]}^p \equiv \sum_{s=0}^m |c_s|^p \int_{\eta_s}^{\eta_{s+1}} \left| \frac{\ddot{\bar{x}}}{g} \right|^p dt = 1 \right\}. \quad (84)$$

For $\delta > 0$ we set

$$M_\delta = \left\{ f \in W_{p,g}^{2,1}[0,1] : \exists \mathbf{c} = \mathbf{c}(f, \delta) \in \mathcal{C} : \left\| \frac{\ddot{f}}{g} - \psi_{\mathbf{c}} \right\|_{L_p[0,1]} \leq \delta \right\}. \quad (85)$$

Let $f \in W_{p,g}^{2,1}[0,1]$. For $1 \leq s \leq m-1$ we set

$$Q_s f(t) = \begin{cases} Q_s^0 f(t), & \eta_s \leq t \leq \eta_{s+1}, \\ (Q_s^0 f)'(\eta_s)(t - \eta_s) + \int_t^{\eta_s} (\tau - t) \ddot{f}(\tau) d\tau, & 0 \leq t \leq \eta_s. \end{cases}$$

For $0 < \gamma_s < \eta_{l_s} - \eta_{l_s-1}$ we set

$$\hat{Q}_s f(t) = Q_s f(t) - \frac{Q_s f(\eta_{l_s} - \gamma_s)}{\eta_{l_{s+1}} - \eta_{l_s} + \gamma_s} (\eta_{l_{s+1}} - t)_+, \quad t \in [\eta_{l_s} - \gamma_s, \quad \eta_{l_{s+1}}].$$

Then $\hat{Q}_s f(\eta_{l_s} - \gamma_s) = 0$.

For $l_s = l_{s+1} - 1$ we denote

$$\varphi_{s+1}^{\gamma_s}(t) = \begin{cases} 0, & 0 \leq t \leq \eta_{l_s} - \gamma_s, \\ t - \eta_{l_s} + \gamma_s, & \eta_{l_s} - \gamma_s \leq t \leq \eta_{l_{s+1}}, \\ \eta_{l_{s+1}} - \eta_{l_s} + \gamma_s, & \eta_{l_{s+1}} \leq t \leq 1. \end{cases}$$

Let

$$\tilde{\varphi}_{s+1} = \begin{cases} \varphi_{s+1}, & l_{s+1} - 1 > l_s, \\ \varphi_{s+1}^{\gamma_s}, & l_{s+1} - 1 = l_s, \end{cases}$$

$$\tilde{Q}_s f(t) = \hat{Q}_s f(t) + \rho_s \tilde{\varphi}_{s+1}(t), \quad t \in [\eta_{l_s} - \gamma_s, \eta_{l_{s+1}} - \gamma_{s+1}], \quad 1 \leq s \leq m-1,$$

$$\tilde{Q}_0 f(t) = Q_0 f(t) + \rho_0 \varphi_1(t), \quad t \in [0, \eta_{l_1} - \gamma_1],$$

$$\tilde{Q}_m f(t) = Q_m f(t) + \rho_m \varphi_{m+1}(t), \quad t \in [\eta_{l_m} + \gamma_{m+1}, 1];$$

here the numbers ρ_s are such that $\tilde{Q}_s f(\eta_{l_{s+1}} - \gamma_{s+1}) = 0$, $0 \leq s \leq m-1$, $\tilde{Q}_m f(\eta_{l_m} + \gamma_{m+1}) = 0$. Denote $Q^* f(t) = -f(t) + f(\eta_{l_m} - \gamma_m) + c(t - \eta_{l_m} + \gamma_m)$, $t \in [\eta_{l_m} - \gamma_m, \eta_{l_m} + \gamma_{m+1}]$, where $c \in \mathbb{R}$ is such that $Q^* f(\eta_{l_m} + \gamma_{m+1}) = 0$;

$$\tilde{Q}f(t) = \begin{cases} \tilde{Q}_0 f(t), & t \in [0, \eta_{l_1} - \gamma_1], \\ \tilde{Q}_s f(t), & t \in [\eta_{l_s} - \gamma_s, \eta_{l_{s+1}} - \gamma_{s+1}], \quad 1 \leq s \leq m-1, \\ Q^* f(t), & t \in [\eta_{l_m} - \gamma_m, \eta_{l_m} + \gamma_{m+1}], \\ \tilde{Q}_m f(t), & t \in [\eta_{l_m} + \gamma_{m+1}, 1]. \end{cases}$$

We set $\tilde{\eta}_{l_s} = \eta_{l_s} - \gamma_s$ for $1 \leq s \leq m$, $\tilde{\eta}_j = \eta_j$ for $j \in \{1, \dots, l_m - 1\} \setminus \{l_1, \dots, l_{m-1}\}$, $\tilde{\eta}_{l_m+1} = \eta_{l_m} + \gamma_{m+1}$, $\tilde{\eta}_j = \eta_{j-1}$, $j \in \{l_m + 2, \dots, n + 1\}$,

$$\tilde{\psi}_j(t) = \begin{cases} 0, & 0 \leq t \leq \tilde{\eta}_{j-1}, \\ t - \tilde{\eta}_{j-1}, & \tilde{\eta}_{j-1} \leq t \leq \tilde{\eta}_j, \\ \tilde{\eta}_j - \tilde{\eta}_{j-1}, & \tilde{\eta}_j \leq t \leq 1, \end{cases} \quad (86)$$

$1 \leq j \leq n + 1$,

$$\tilde{L} = \left\{ \sum_{j=1}^{n+1} c_j \tilde{\psi}_j : c_j \in \mathbb{R} \right\}. \quad (87)$$

Arguing similarly as in [30], we obtain the following assertions.

Lemma 10. *The inclusion $f + \tilde{Q}f \in \tilde{L}$ holds.*

Lemma 11. *There are numbers $\delta > 0$, $\gamma_0 > 0$, and $C_* = C_*(g, v, p, q, \bar{x}, \bar{y})$, for which, for any $\gamma \in (0, \gamma_0)$, there are $\gamma_1, \dots, \gamma_{m+1} \in (0, \gamma)$ such that*

$$\int_0^1 v^q(t) |Q_L f(t)|^q dt - \int_0^1 v^q(t) |\tilde{Q}f(t)|^q dt \geq C_* \gamma$$

for any $f \in M_\delta$.

5 The strict decreasing of widths

Let $n \in \mathbb{Z}_+$. We prove that

$$d_{n+1}(W_{p,g}^{r,k}[0, 1], L_{q,v}[0, 1]) < d_n(W_{p,g}^{r,k}[0, 1], L_{q,v}[0, 1]). \quad (88)$$

Let $\bar{\theta} = \bar{\theta}_n$, $(\bar{x}, \bar{y}, \bar{\theta}) \in SP_n$.

For $r = 2$ we define the set Λ by (64).

Consider two cases.

1. $r = 2$, $\Lambda \neq \emptyset$,
2. either $r = 2$ and $\Lambda = \emptyset$ or $r \geq 3$.

In the first case we choose δ according to Lemma 11 and take $M = M_\delta$ as defined in (85). In the second case we set

$$M = \left\{ f \in W_{p,g}^{r,k}[0, 1] : \left\| \frac{f^{(r)}}{g} - \frac{\bar{x}^{(r)}}{g} \right\|_{L_p[0,1]} \leq \delta \quad \text{or} \quad \left\| \frac{f^{(r)}}{g} + \frac{\bar{x}^{(r)}}{g} \right\|_{L_p[0,1]} \leq \delta \right\}.$$

For $r = 2$ we define the function $P_L f$ by formula (67). For $r \geq 3$ we set

$$P_L x(t) := x(t) - (-1)^{r-k} \int_0^1 G(t, \tau, \xi, \eta) x^{(r)}(\tau) d\tau = \sum_{j=1}^n c_j(x) H(t, \eta_j)$$

(see (31)).

Let $N = W_{p,g}^{r,k}[0, 1] \setminus M$. We show that there exists $\sigma > 0$ such that

$$\|f - P_L f\|_{L_{q,v}[0,1]} \leq \bar{\theta}^{-1} - \sigma, \quad f \in N. \quad (89)$$

Suppose the contrary. Similarly as in [30, p. 389] it can be proved that there exists a function x such that $\|x - P_L x\|_{L_{q,v}[0,1]} = \bar{\theta}^{-1}$, $\left\| \frac{x^{(r)}}{g} \right\|_{L_p[0,1]} = 1$; in addition, in case 1 the equality $\frac{\ddot{x}}{g} = \psi_{\mathbf{c}}$ cannot hold for any $\mathbf{c} \in \mathcal{C}$ (see (84)), and in case 2 $x^{(r)} \neq \pm \bar{x}^{(r)}$. Hence, there exists $\tau_* \in (0, 1)$ such that $\frac{x^{(r)}}{\bar{x}^{(r)}} \Big|_{(\tau_* - \varepsilon, \tau_* + \varepsilon)} \neq \text{const}$ for any $\varepsilon > 0$; moreover, $\tau_* \notin \{\eta_j : j \in \Lambda\}$ in case 1. By Corollary 1, there exist intervals Δ' , $\Delta'' \subset (0, 1)$ such that $G(t, \tau, \xi, \eta) \neq 0$ for a.e. $(t, \tau) \in \Delta' \times \Delta''$ and $\frac{x^{(r)}}{\bar{x}^{(r)}} \Big|_{\Delta''} \neq \text{const}$. By (32) and (33),

$$(-1)^{r-k} \frac{G(t, \tau, \xi, \eta) \bar{x}^{(r)}(\tau)}{\bar{x}(t)} \geq 0, \quad \int_0^1 \frac{G(t, \tau, \xi, \eta) \bar{x}^{(r)}(\tau)}{\bar{x}(t)} d\tau = (-1)^{r-k}.$$

Arguing as in [30, p. 389–390], we lead to a contradiction. This completes the proof of (89).

We construct a subspace \tilde{L} of dimension $n + 1$ such that

$$\sup_{x \in W_{p,g}^{r,k}[0,1]} \inf_{z \in \tilde{L}} \|x - z\|_{L_{q,v}[0,1]} < \bar{\theta}^{-1}. \quad (90)$$

This implies (88).

Consider case 1. Define the functions $\tilde{\psi}_j$ and the space \tilde{L} by formulas (86) and (87) correspondingly. We choose $\delta > 0$, $\gamma_0 > 0$, and for arbitrary $\gamma \in (0, \gamma_0)$ we find $\gamma_1, \dots, \gamma_{m+1}$ such that the following inequality holds by Lemma 11:

$$\int_0^1 v^q(t) |Q_L x(t)|^q dt - \int_0^1 v^q(t) |\tilde{Q}x(t)|^q dt \geq C_* \gamma, \quad x \in M, \quad \gamma \in (0, \gamma_0). \quad (91)$$

For $x \in N$ we define the function $\tilde{P}x = \tilde{P}_\gamma x$ as follows: if $P_L x = \sum_{j=1}^n c_j(x) \psi_j$, then

we set $\tilde{P}x = \sum_{j=1}^{l_m} c_j(x) \tilde{\psi}_j + \sum_{j=l_m+2}^{n+1} c_{j-1}(x) \tilde{\psi}_j$. By assertion 2 of Lemma 4 and (70), the set $\{c_i(x) : x \in W_{p,g}^{2,1}[0, 1], \quad 1 \leq i \leq n\}$ is bounded; hence,

$$\sup_{x \in N} \|\tilde{P}_\gamma x - P_L x\|_{L_{q,v}[0,1]} \xrightarrow{\gamma \rightarrow 0} 0.$$

Therefore, if γ is sufficiently small, then $\|x - \tilde{P}x\|_{L_{q,v}[0,1]} \stackrel{(89)}{\leq} \bar{\theta}^{-1} - \frac{\sigma}{2}$ for any $x \in N$. Since $x + \tilde{Q}x \in \tilde{L}$ by Lemma 10, this together with (91) implies (90).

In case 2 the arguments are similar as in [30, p. 391].

6 Applications

In [31] order estimates for Kolmogorov widths of classes $W_{p,g}^{r,k}[0, e^{-1}]$ were obtained, where

$$g(x) = x^{-\beta_g} |\ln x|^{-\alpha_g} \rho_g(|\ln x|), \quad v(x) = x^{-\beta_v} |\ln x|^{-\alpha_v} \rho_v(|\ln x|), \quad (92)$$

$$\beta_g + \beta_v = r + \frac{1}{q} - \frac{1}{p}, \quad \beta_v \notin \left\{ \frac{1}{q}, \frac{1}{q} + 1, \dots, \frac{1}{q} + r - 1 \right\}, \quad (93)$$

$$\alpha_g + \alpha_v > \left(\frac{1}{q} - \frac{1}{p} \right)_+, \quad (94)$$

ρ_g, ρ_v were absolutely continuous functions such that

$$\lim_{y \rightarrow +\infty} \frac{y \rho'_g(y)}{\rho_g(y)} = \lim_{y \rightarrow +\infty} \frac{y \rho'_v(y)}{\rho_v(y)} = 0. \quad (95)$$

Denote $\alpha = \alpha_g + \alpha_v$, $\rho(y) = \rho_g(y) \rho_v(y)$.

Theorem 2. Let $r \in \mathbb{N}$, $1 < q \leq p < \infty$, let conditions (92), (93), (94), (95) hold, where $\beta_v \in \left(\frac{1}{q} + k - 1, \frac{1}{q} + k\right)$, $1 \leq k \leq r - 1$. Let $\bar{\theta}_n = \sup sp_n$, where the set sp_n is defined according to (7). Then

$$\bar{\theta}_n \asymp \begin{cases} n^r, & \alpha > r + \frac{1}{q} - \frac{1}{p}, \\ n^r (\log n)^{-r - \frac{1}{q} + \frac{1}{p}}, & \alpha > r + \frac{1}{q} - \frac{1}{p}, \quad \rho_g \equiv 1, \quad \rho_v \equiv 1, \\ n^{\alpha - \frac{1}{q} + \frac{1}{p}} [\rho(n)]^{-1}, & \alpha < r + \frac{1}{q} - \frac{1}{p}. \end{cases} \quad (96)$$

Moreover, if α and ρ are such that $gv \in L_{\varkappa}[0, e^{-1}]$ with $\frac{1}{\varkappa} = r + \frac{1}{q} - \frac{1}{p}$, then

$$\lim_{n \rightarrow \infty} n^r \theta_n^{-1} = \lambda_{rqp}^{-1} \|gv\|_{L_{\varkappa}[0, e^{-1}]}, \quad (97)$$

where λ_{rqp} is the first eigenvalue for the problem

$$(-1)^{r+1} ((x^{(r)})_{(p)})^{(r)} + \lambda^q x_{(q)} = 0$$

with periodic boundary conditions.

Proof. By Theorem 1,

$$\theta_n^{-1} = d_n(W_{p,g}^{r,k}[0, e^{-1}]).$$

This together with [31, Theorem 1, Examples 1–3] yields (96).

Let us prove (97). To this end we check that

$$\|\tilde{I}_{r,g,v}^{a,b,k} \varphi\|_{L_q[a,b]} \lesssim_{g,v,p,q,r} \|gv\|_{L_{\varkappa}[a,b]} \|\varphi\|_{L_p[a,b]}. \quad (98)$$

If $a \geq \frac{b}{2}$, then (98) follows from [31, Proposition 2]. Let $a < \frac{b}{2}$. Then from [31, Propositions 3, 4] it follows that

$$\|\tilde{I}_{r,g,v}^{a,b,k} \varphi\|_{L_q[a,b]} \lesssim_{g,v,p,q,r} \left(\int_{|\log b|}^{|\log a|} t^{-\frac{\alpha pq}{p-q}} [\rho(t)]^{\frac{pq}{p-q}} dt \right)^{\frac{1}{q} - \frac{1}{p}} \quad (99)$$

(for $p = q$ we take the norm in L_{∞}). On the other hand,

$$\|gv\|_{L_{\varkappa}[a,b]} \|\varphi\|_{L_p[a,b]} = \left(\int_{|\log b|}^{|\log a|} t^{-\alpha \varkappa} [\rho(t)]^{\varkappa} dt \right)^{\frac{1}{\varkappa}}. \quad (100)$$

From the inequality $\varkappa < \frac{pq}{p-q}$ it follows that $\|(x_n)_{n \in \mathbb{N}}\|_{l_{\frac{pq}{p-q}}} \leq \|(x_n)_{n \in \mathbb{N}}\|_{l_{\varkappa}}$ for any sequence $(x_n)_{n \in \mathbb{N}}$. Writing the integrals (99) and (100) in terms of sums of integrals over intervals of length $l \in [1, 2]$, we obtain (98).

In order to complete the proof of (97) we argue similarly as in [32, §4]: we apply the Buslaev's result [4] for piecewise-continuous weights and pass to limit. \square

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